Applying pure mathematics: mathematics in the natural sciences

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'The book of nature is written in the language of mathematics.' This quotation, attributed to Galileo, seemed to hold to an *unreasonable*[1] extent in the era of quantum mechanics. However, as the epicentre of natural science has moved towards biology, one wonders if it is now dated.

Meanwhile, mathematics has developed in two different ways – so-called applied mathematics addresses problems that are known to be related to the natural sciences and engineering. Meanwhile (pure) mathematics has developed primarily based on its own internal criteria and processes.

It is thus natural to wonder what role (pure) mathematics will continue to play in the natural sciences. As has been observed by Gian–Carlo Rota, among others, it is the *concepts*, not the results, of mathematics that have proved to be of use. We remark that applications in engineering (for example, cryptography) are often of a very different nature; we shall not consider such applications here.

Here we shall outline a typical instance of an application of pure mathematics in natural science – the use of the concept of curvature in the *general theory of relativity*, and comment on whether and how similar applications are likely in the future. We begin with some general remarks.

Natural science as an inverse problem

According to the scientific method, one accepts a theory if it can make falsifiable predictions which are in agreement with observations. While the predictions of a theory can be obtained by deductive reasoning, to arrive at a theory one must use some guesswork. Thus, discovery in natural science is what is called an inverse problem.

It is illustrative to consider another complex inverse problems – vision. While our eyes see two-dimensional images, to make sense of the world we need to resolve it into different objects. Given the positions of objects and the nature of incident light, it is relatively straightforward to deduce how the image should look. However, the problem of resolving the world into objects and deducing their properties from the images we see is not even solvable in principle. In practice, we solve this by making many simplifying assumptions. Indeed, optical illusions are based on creating images violating these assumptions.

A theory in natural science is analogous to a description of the positions and properties of various objects in our sight. One can deduce from the positions and properties of the objects a prediction of the images on our retinas, allowing us to test the 'theory'.

Our visual cortex deduces the positions and properties of the objects from the image they produce. While this is a remarkable feat, we can at least start with our knowledge of different kinds of objects around us and the range of their properties.

In natural science, we do not even have such a list of the analogues of objects, namely concepts. Our brains, designed by natural selection, can conceive only of a limited range of concepts. Indeed textbooks of quantum mechanics typically begin with a list of apparent paradoxes, which are consequences of an implicit assumption that an electron is either a particle or a wave.

Mathematics, through its processes of abstraction, generalizations and extensions, allows us to arrive at concepts that go beyond the untrained imagination. These become candidates for the concepts used in a theory.

We shall illustrate this by sketching briefly the concept of *intrinsic curvature*, its history and its role in the general theory of relativity. As with many useful mathematical concepts, its construction is not particularly complex or technically difficult. Moreover, it is virtually certain that any reasonably advanced mathematics would include this concept. Nevertheless, it seems unlikely that, for instance, an experimental approach to science would uncover this concept.

Extrinsic and intrinsic curvature

The curvature of a circle clearly decreases as its radius increases. Thus, it is natural to take the curvature of a circle of radius r to be the reciprocal $1/r$ of its radius. We extend this to define the curvature at a point P on a curve C in the plane just like tangents are defined in Calculus – we take two points Q and R on the curve C close to P and consider the reciprocal $1/r(P, Q, R)$ of the radius $r(P, Q, R)$ of the circle through P, Q and R (we set $1/r(P, Q, R) = 0$ if the points P, Q and R are co-linear). As Q and R approach P, we get a limit, which we define to be the curvature.

The case of surfaces is a little more complicated. While a sphere is clearly curved by the same amount at all points and directions, a cylinder is curved in one direction but not the other. Thus, the curvature of a surface depends not only on a point on the surface but also a (tangent) direction at that point. Euler observed that the curvatures in various directions are determined by two numbers – the *principal curvatures*. This is analogous to an ellipse being determined by its major and minor axes – indeed, both cases are governed by a quadratic function, which can be simplified by completing squares. Both the principal curvatures at any point of a sphere of radius r are $1/r$ while for a cylinder of radius r, the principal curvatures are $1/r$ and 0. The principal curvatures of a plane are 0.

Geometric properties of a surface can be classified as extrinsic or intrinsic. An intrinsic property of a surface is one that depends only on distances measured along the surface, i.e. is unchanged on bending the surface. Thus, a square can be bent into a piece of a cylinder, so intrinsic properties of the square and the corresponding piece of the cylinder must coincide.

Intrinsic differential geometry, the study of intrinsic properties, was born with the work of Gauss, who observed that we cannot make a map of a region of a sphere without distorting distances. Thus, while we can wrap a piece of paper around a cylinder smoothly, we cannot do so on a sphere without introducing wrinkles. The fundamental discovery of Gauss was that the product of the principal curvatures, which we now call the Gaussian curvature, depended only on distances measured along the surface, i.e. an intrinsic property of the surface.

Note that the principal curvatures themselves are not intrinsic (consider for example, a square and a piece of the cylinder). Hence, while the Gaussian curvature is intrinsic, it is defined in terms of quantities that are not. This mathematically unsatisfactory situation was remedied by Riemann, who developed a genuinely intrinsic framework for differential geometry, i.e. one where we consider just a surface and the distances between points on it obtained by measured along the surface. In the process, Riemann defined intrinsic curvatures for what we call Riemannian manifolds, which are the natural higher dimensional analogues of surfaces with distances coming from space.

It is not difficult to see that a sphere is intrinsically curved – for example, the area $A(r)$ of a disc or radius r on the sphere is less than the area πr^2 of the disc in the plane of the same radius. Indeed a Taylor expansion of $A(r)$ near $r = 0$ allows us to quantify curvature. However, rather than considering just the magnitudes of curvature, it is more useful to consider a more complicated object called the Riemann curvature tensor.

Curvature and relativity

In Einstein's general relativity, gravity is due to the intrinsic curvature of space. While the curvature of space seems to most people to be a bizarre notion, there is no better reason for space to be flat (i.e. not curved) than for the earth to be flat (as was once believed to be the case), especially once one accepts that there is no absolute medium called space but only distances between events.

What makes (intrinsically) curved space much harder to accept than the round earth is that while the (extrinsic) curvature of objects is very familiar, and we merely have to realize that this applies to the earth, intrinsic curvature has no significance in our everyday life and is hence an alien concept.

Merely accepting that space could be intrinsically curved is of little use in developing relativity – one needs a precise and quantitative concept of intrinsic curvature. It is this that developed through the natural evolution of mathematics.

Mechanics can be formulated in terms of the principle of least action (generalizing the principle that light travels by the fastest path). Einstein formulated general relativity in terms of an action that depended on curvature – specifically Ricci curvature of a Lorentzian manifold. Such a formulation made use of not only Riemann's curvature tensor, but the work of several mathematicians that went beyond this.

Mathematics as a genetic algorithm

Mathematics develops through a process somewhat analogous to Darwinian evolution, with new results being extensions of older results, sometimes adapted to the external environment. Unlike evolution, this is not based on random variation but on partially systematic rules for modification and extension. With most mathematics forgotten soon after its birth and some pieces of work flowering into new fields, there is also a selection process; as with life this is partly based on fitness and partly random. Further, the insistence on rigour serves as a filter which ensures most mutants do not survive – a necessity for any fruitful process of evolution.

Result of this process is a proliferation of new concepts, most of which are not of any use outside mathematics, but some of which provide the crucial concepts for the development of natural science, as we have illustrated in the case of curvature. As with the case of life, the most useful steps are the major transitions, analogous to the invention of the Eukaryotic cell or multi-cellularity. But these cannot be isolated from the continuous process of evolution.

It is hard to imagine any of these concepts being developed other than through the internal development of mathematics or some similar system (for example, in many ways statistical physics develops like mathematics). It takes successive steps of refinement over the course of time to arrive at the right concept, so one cannot expect to provide it on demand. On the other hand, simply trying to invent concepts will clearly lead to a deluge of silliness.

A defence of mathematics?

We have argued that mathematics provides an unparallelled process for generating new concepts, and such concepts are essential for natural science as discovery is an inductive (not deductive) process. However, there are at least two kinds of barriers to this synergy.

Like evolution, a rerun of the development of mathematics will lead to a different development. Just as one can argue that any sufficiently developed process of natural selection will lead to the invention of the eye, many concepts such as curvature are bound to be present in sufficiently developed mathematics. On the other hand, being governed in practice to a large extent by social factors, the development can be dominated by the analogues of the financial bubbles, taking away resources from what would be more fruitful developments. A historically striking case of this is that, at least till recently, probability has not been regarded as in the mainstream of mathematics in the West (though it has in Russia). From the point of view of natural science, the greater value attached to number theoretic rather than probabilistic concepts is clearly unfortunate. Ironically, in the Anglo-Saxon world probability was also neglected by applied mathematicians.

The second barrier is that with the explosion of science, it is not practical for scientists to learn mathematics far from that which is already known to be applicable to their field. In the other direction, there are also cultural barriers to a mathematician venturing into natural science. Rigour and depth are central to mathematics. However, as a mathematical framework for most natural science is only approximate, a rigorous deduction of the consequences of the framework is of little value. Further, use of deep mathematics generally means that one is using a long chain of logical deductions from a good approximation to reality – often giving a poor approximation.

Thus, while one may argue that mathematics can continue to play a significant role in natural science, whether it will is another question.

1. Wigner, E., *Commun. Pure Appl. Math.*, 1960, **13**, 114.

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