On the gravitational deflection of light and particles

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The gravitational deflection of light deduced from applying the equivalence principle is stated to be only half the general relativistic result. I show that the correct result is obtained entirely from applying the equivalence principle if the wave nature of light and particles is taken into account. This is equivalent to incorporating the gravitational redshift in the calculation. In Einstein's 1911 derivation, he used the wave aspect to get half the general relativistic value, but perhaps presumed that it was physically identical to the part that came from the application of the equivalence principle to the ray trajectory. Here I point out that the two contributions are independent and have different physical origin and characteristics. Adding the two contributions together gives the correct general relativistic value for light and particles. For material particles, the 'particle' part dominates, and for relativistic quantum particles, the deflection approaches that for light.

THE purpose of this paper is to explore a conceptual issue connecting general relativity and the wave nature of particles and light. I will show that the full general relativistic result for the deflection of light and particles follows from the equivalence principle alone when applied consistently to both the particle and wave aspects. The total deflection is the sum of the Newtonian deflection of the mean trajectory and the gradual bending of the wavefronts due to the gravitational redshift, and both these follow from the equivalence principle. The result stresses the fact that the physical results of general relativity are potentially fully contained in the physics of Lorentz transformations combined with the equivalence principle. Though this is known, since general relativity is derived using special relativity and equivalence principle as the guiding principles, it is not adequately appreciated. Also, this result refutes the standard claim that it is not possible to obtain the correct result for gravitational deflection from equivalence principle alone. Further, the identification of the gravitational deflection of light as the crucial test of general relativity, in contrast to the gravitational redshift, does not seem defendable.

The gravitational deflection of light near a massive body of mass M was derived by Einstein in his 1911 precursor paper[1] to General Theory of Relativity as

$$\alpha = \frac{2GM}{c^2 R}, \tag{1}$$

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R is the impact parameter, and it is equal to the radius of the body for grazing incidence. This is half the correct general relativistic value. The full general relativistic result[2] for the deflection angle is

$$\alpha = \frac{4GM}{c^2 R}. \tag{2}$$

The relevant quantities are marked in Figure 1.

An elementary derivation of the expression for the deflection of light within Newtonian mechanics is possible, and it gives half the general relativistic value[3,4]. The derivation is the same for a fast material particle, and for light considered as a particle. Essentially one calculates the Newtonian bending of the trajectory, either assuming it to be corresponding to that of a freely falling particle observed from a stationary frame, or that of a free particle observed from a uniformly accelerating laboratory (this derivation is different from Einstein's original derivation). To get an approximate estimate consider the particle, or light coupled to gravity, that will fall through a distance

$$s = \frac{gt^2}{2}, \tag{3}$$

during propagation close to the massive object. t is approximately $2R/v$, where R is the radius of the massive body and v is the velocity of the fast particle. Identical result is obtained if we use the equivalence principle. In a frame accelerating up with acceleration $-g$, the coordinate deflection of *any straight trajectory* over time t is $s = gt^2/2$.

The angle of deflection when $gt^2 \ll vt$ is the angle between the asymptotes. Since the slope of the parabola resulting from the free fall is gt, the outgoing asymptote intersect the incoming straight line trajectory at $t_0 = t/2$. The time duration of deflection is $t \simeq 2R/v$. Therefore the angle of deflection is

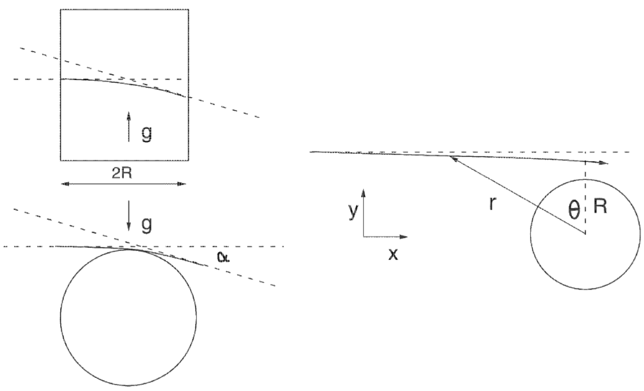

Figure 1. *Left*, Correspondence between the gravitational deflection in an accelerating frame and in a gravitational field. *Right*, Geometrical relations used in the calculation of the bending of the trajectory.

$$\alpha \simeq (gt^2/2)/(vt/2) = \frac{gt}{v} = \frac{2GM}{v^2R}. \quad (4)$$

This approximate result assumed that most of the deflection happens close to the massive body.

The same result is obtained from a more complete and correct calculation. The angle of deflection is simply the slope of the trajectory as $t \to \infty$, relative to the slope at $t \to -\infty$. If the trajectory is denoted as $\vec{S}(t)$,

$$\frac{d\vec{S}}{dt} = \vec{v} = v_x\hat{x} + v_y\hat{y}. \quad (5)$$

Referring to Figure 1, at $t \to -\infty$, $\vec{v} = v_x\hat{x}$. The angle of deflection is

$$\alpha = \frac{v_y}{v_x}. \quad (6)$$

(This reduces to gt/v for the approximate situation considered earlier). For a proper calculation we need to estimate v_y accumulated over the entire trajectory from $t \to -\infty$ to $t \to +\infty$. Applying the equivalence principle to any ray trajectory that corresponds to the propagation at velocity v, the apparent deflection as observed from an accelerating frame is

$$\alpha = \frac{\delta v_\perp}{v}, \quad (7)$$

where $\delta v_\perp = v_y$ is the change in the transverse velocity. Initially, $v = v_x$. The y-component of the gravitational acceleration, $g_y = g\cos\theta$ (see Figure 1). To apply the equivalence principle we set the acceleration, $a_y(r) = -g\cos\theta = GM\cos\theta/r^2$.

$$v_y = \int_{-\infty}^{+\infty} a_y dt = \int_{-\infty}^{+\infty} a_y \frac{dt}{dx} dx = \frac{2GM}{v}\int_0^{+\infty} \frac{\cos\theta\, dx}{r^2}$$
$$= \frac{2GM}{v}\int_0^{+\infty} \frac{y dx}{(x^2+y^2)^{3/2}} = \frac{2GM}{vy}. \quad (8)$$

Since y is the nearly constant impact parameter (which we write now as R), the deflection is

$$\alpha = \frac{v_y}{v} = \frac{2GM}{v^2R} = \frac{2GM}{c^2R}\left(\frac{c^2}{v^2}\right). \quad (9)$$

Since we used the equivalence principle and invoked the accelerating frame, this derivation is valid for light as well, provided we assume a finite velocity for light. The deflection of a relativistic particle is close to that of light, but for the material particle the deflection is larger by factor c^2/v^2. Important point to note is that we have treated light as a ray, or as a particle, and its wave nature was not used. We did not even have to use the equivalence principle if we just assumed that light was a particle with an equivalent mass of E/c^2. (In such case, we have to integrate the expression $\int_{-\infty}^{+\infty} F_y dt$ to find the change in transverse momentum, and then divide by the longitudinal momentum to get the deflection[3]. The final expressions are the same as in eq. (8).) This is a doubly Newtonian derivation – Newtonian gravity and Newtonian light give the deflection as $\alpha = 2GM/c^2R$. This will happen even if there is no relativistic effect of order v^2/c^2, like time dilation and gravitational redshift. In other words, Newtonian gravity or Galilean Equivalence principle, and the assumption of finite (not necessarily invariant) velocity of light are sufficient to derive this expression for deflection.

The difference between applying the equivalence principle, and integrating the expression $\int_{-\infty}^{+\infty} F_y dt$ containing the gravitational force is that the latter method is not directly applicable to light without assuming an equivalent passive gravitational mass for the photon. The derivation using the accelerated frame is valid for any trajectory that propagates at velocity v.

Now we come to the crucial point of this discussion. If we take into account the wave nature of light, there is an *additional* contribution coming from the time dilation in relativity. The observation that this contribution is independent and additional to the Newtonian deflection of the mean trajectory is the main point of the this paper. This contribution comes about as follows.

Consider plane waves propagating freely in a straight line. We can also consider a more realistic spatially Gaussian wave-packet propagating near a massive body of mass M and radius R. Without using any aspect of special relativity, like time dilation, we see from the equivalence principle that the ray of propagation (any normal to the wavefronts) deflects and that this deflection is physically and mathematically same as the one expressed in eqs (8) and (9). But this does not yet use the relativistic result that the rates of clocks are different at different heights in a uniform gravitational field. Once we also take into account the gravitational redshift, there is *additional* bending of light.

The gravitational redshift formula can be derived in many ways[2]. One way is to use special relativity and the equivalence principle. Another simpler way is to use the conservation of energy in a gravitational field, along with the results $E = mc^2$, and also $E = h\nu$. Either way, we get that the frequency at a distance h closer to the body is higher by

$$\nu' = \nu(1 + gh/c^2)$$

$$\Delta\nu = \nu' - \nu = \nu gh/c^2. \quad (10)$$

There is no need to use any specific general relativistic aspect for this derivation and hence this is one of the most straightforward results from combining the equivalence principle and special relativity, with the fundamental quantum equation for energy (It is sufficient to use the quantum equation for energy and the conservation of energy).

Variation of the redshift factor as the height changes in the gravitational field gives rise to bending of *waves of light*. In fact, this is the physical reasoning used by Einstein in his 1911 derivation of the bending of light, and not the reasoning using accelerated frames and equivalence principle. Instead of describing the effect as due to redshift, he considered the equivalent effect that the velocity of light varies with height in a gravitational field, resulting in the bending of wavefronts[1]. Since the velocity of the wavefronts in a higher negative potential is smaller, the wavefronts bend towards the source of the gravitational field. Using the redshift concept, the wave fronts (surfaces of equal phase) are closer together in a deeper potential, as shown in the Figure 2, and they gradually bend. First we indicate the result with an approximate derivation. The deflection over the duration $t = 2R/v$, where v is the phase velocity of the waves, is given by

$$\delta = (\Delta\nu)t.\lambda_0 / h = \frac{\nu_0 g}{c^2}\lambda_0 \frac{2R}{v} = \frac{2GM}{c^2 R}, \qquad (11)$$

λ_0 and ν_0 are the mean wavelength and frequency and $\lambda_0\nu_0 = v$. This is the accumulated path length difference at two points separated by distance h, divided by h. A more complete calculation that gives the same result is done later when the deflection of matter waves is considered. The most important aspect of this part of the deflection is that it is *independent of the velocity of the waves*, unlike the deflection we obtained applying the equivalence principle to the particle or ray. This crucial difference immediately reveals that the two contributions are of different physical origin, and are independent.

From the history of the problem it seems to me that after Einstein derived the deflection of light in his 1911 paper, he assumed that the result he obtained considering the variation of the velocity of light in a gravitational field

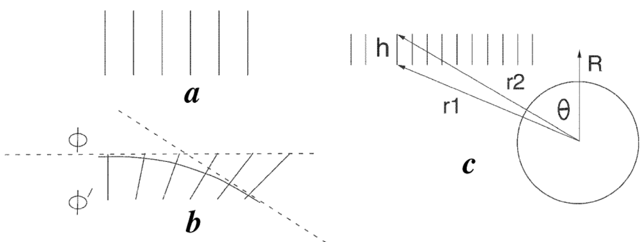

Figure 2. Gradual bending of the wavefronts due to the gravitational redshift factor affecting the wavelength at different heights in a gravitational field. ***a***, Plane wavefronts, with spacing indicating the wavelength; ***b***, Gradual bending of the normal to the wavefront consistent with gravitational redshift, and ***c***, The geometrical relations needed to derive the bending of the wavefronts in the gravitational field. The quantity h is greatly exaggerated for clarity.

and the wave aspect of light was physically the same as the one from applying the Newtonian deflection or the equivalence principle to a ray of light or a particle[1]. From the character of the two contributions it is clear that these are physically different. Only for light in vacuum the two contributions have the same magnitude, and since the 1911 derivation was done for this case, it seems clear why it was not realized that the two contributions are physically independent and different. The concept of a matter wave was introduced only after 1920. But if the calculation in the 1911 paper is repeated for matter waves, with average group velocity v of the particle, it would immediately be clear that the resulting deflection is very different from the deflection derived from applying the equivalence principle to the same particle. I illustrate this in the next paragraph. Also, if the calculation in the 1911 paper is done for sound waves, or other 'material waves', the presence of two independent effects is revealed. So, the full physical aspects needed to derive the correct deflection of light was already contained in the 1911 paper itself. One does not need full general relativity for deriving the expression for the deflection of light. What is needed is the equivalence principle, conservation of energy, and the wave nature of light. One part of the deflection comes from the free fall of the particle or light ray in the gravitational field, and the deflection depends on the average velocity of the test particle. The other part comes from the redshift factor. This is always given by $\delta = 2GM/c^2R$, independent of the velocity. Combining the two we are able to get the correct formula for the gravitational deflection. It is the 'relativistic' part, affecting the waves, that is independent of the velocity of the waves, and the Newtonian contribution is proportional to the inverse of the square of the velocity. For light, both contributions have the same magnitude, and they add to give the full deflection,

$$\alpha_{\text{light}} = \frac{2GM}{v_{\text{light}}^2 R} + (\Delta\nu)t\cdot\lambda_0 / h = \frac{4GM}{c^2 R}, \qquad (12)$$

using eqs (9) and (11).

For deriving the expression for the deflection of quantum particles, consider the matter wave propagating in a gravitational field (the derivation is applicable to any wave). The matter wave obeys the same equation (eq. 10) of frequency redshift as for light,

$$\nu' = \nu(1 + gh/c^2).$$

The angle of deflection can be calculated by integrating the phase over the entire trajectory to find the path difference between two points on the wavefront separated by a distance h and then dividing by h. Referring to the part C of Figure 2, and noting that h or $|\vec{r1} - \vec{r2}| \ll R$, and that the angle θ is essentially the same for both the vectors $\vec{r1}$ and $\vec{r2}$, the phase difference is

$$\Delta\varphi = \int_{-\infty}^{+\infty} \omega(r1)\mathrm{d}t - \int_{-\infty}^{+\infty} \omega(r2)\mathrm{d}t$$

$$= \frac{-GM\omega_0}{c^2} \int_{-\infty}^{+\infty} \left(\frac{1}{r1} - \frac{1}{r2} \right) \frac{\mathrm{d}t}{\mathrm{d}x} \mathrm{d}x = \frac{-2GM\omega_0}{c^2} \int_0^{+\infty} \frac{\delta r}{r^2} \frac{1}{v} \mathrm{d}x$$

$$= \frac{-2GM\omega_0}{c^2} \int_0^{+\infty} \frac{h\cos\theta}{r^2} \frac{\mathrm{d}x}{v}$$

$$= \frac{-2GM\omega_0}{c^2} \int_0^{+\infty} \frac{hy}{(x^2 + y^2)^{3/2}} \frac{\mathrm{d}x}{v} = \frac{-2GM\omega_0 h}{c^2 yv}, \tag{13}$$

where $h = |\vec{r1} - \vec{r2}|$, a constant for small deflection. ω_0 is the mean frequency. The negative phase difference indicates bending towards the source.

The difference in path length over the distance h is $\Delta l = \Delta\varphi \cdot \lambda_0 / 2\pi$. Denoting y as R, the angle of deflection is

$$\alpha_{\text{wave}} = \frac{\Delta\varphi\lambda_0 / 2\pi}{h} = \frac{2GM\omega_0\lambda_0}{2\pi c^2 Rv} = \frac{2GM}{c^2 R}, \tag{14}$$

since $\omega_0\lambda_0/2\pi = v$, the phase velocity of the waves. Since the phase velocity has dropped out of the final expression, the deflection is same for waves with different de Broglie wavelength (as one would expect from the consistency with the equivalence principle) and the derivation is applicable for a general wave-packet solution of the Schrödinger equation. This is important, since the achromaticity of the gravitational bending makes the formula universal and applicable to any type of wave with any wavelength.

The deflection α_{wave} is negligible compared to the trajectory deflection, eq. (9), for moderate speeds by a factor v^2/c^2. This highlights the fact that there are in fact two physically distinct contributions. The total deflection of fast material particles is the sum of the two contributions, eqs (9) and (11),

$$\alpha_g = \frac{2GM}{c^2 R} \left(1 + \frac{c^2}{v^2} \right). \tag{15}$$

Thus, Einstein's derivation of 1911 gives only the velocity independent part of total deflection and misses out the larger contribution when applied to waves with velocity different from that of light. As the particle becomes relativistic, the gravitational deflection approaches that of light, and becomes exactly

$$\alpha_g = \frac{4GM}{c^2 R}, \tag{16}$$

for light.

An interesting case is the deflection of sound waves in a gravitational field (what I consider is the deflection of sound around an isolated spherical object of mass M and radius R. This way, asymptotes can be defined accurately). If we repeat the calculation by Einstein in his 1911 paper considering sound waves instead of light waves, or if we follow the derivation using the redshift factor in the gravitational field as we did for the case of matter waves, the deflection is

$$\alpha = \frac{2GM}{c^2 R}. \tag{17}$$

This is independent of the velocity of the sound waves! The equivalence principle applied to the situation without bothering about the gravitational redshift along the vertical in the wavefront will give an additional deflection due to the fact that the density of the medium varies with the vertical distance such that the equivalent gravitational force is balanced by the hydrostatic pressure. The density of the medium reduces with height in the gravitational field, and this contribution to the deflection depends on the properties of the medium, the gravitational acceleration, and the velocity of sound (In the atmosphere, the velocity of sound is determined more by the variation of temperature and the velocity in fact decreases with height). Denoting this part of the deflection as α_{med}, the total deflection is

$$\alpha_t = \frac{2GM}{c^2 R} + \alpha_{\text{med}}. \tag{18}$$

Again, as in the case of matter waves, we see that one part of the deflection comes from the gravitational redshift along a direction parallel to $\vec{g}$, and it is independent of the velocity of the waves. (Since sound waves require the medium, and since the medium is 'static' in a gravitational field, deflection of propagating sound energy is different from that of propagating entities that do not require a medium. From physical considerations, the velocity independent term $2GM/c^2R$ is necessarily present for consistency with relativity, irrespective of whether there is a velocity dependent Newtonian term).

As another instructive application let us consider the gravitational bending of light in a gravitational field as it propagates in a medium. The velocity is reduced by the refractive index factor n. Essentially light spends more time in the gravitational field and therefore more bending is expected from Newtonian considerations. But the bending angle α is not $4GMn^2/c^2R$, where R is the spatial region over which gravity acts. It is only the Newtonian part that depends on the velocity that is increased by the refractive index factor, and the contribution from the gravitational redshift remains unaffected. Thus the equation for bending of light should be

$$\alpha = \frac{2GM}{c^2 R}(1 + n^2). \tag{19}$$

(This is written for a situation where the light comes from a region far away compared to the size of the massive object, such that asymptotes can be defined).

Similar velocity independent terms are present in other situations involving gravitational fields and matter or light waves, as highlighted in ref. 5 in the context of the Sagnac effect. In the context of the discussion in that paper, the relativistic part of the gravitational deflection is in fact the ratio of the local gravitational potential (generated by the massive body) and the gravitational potential due to all the masses in the universe. For a universe with critical density ($\rho \simeq 2 \times 10^{-29}$ g/cm^3) and an effective causal size of $R_H \simeq 10^{28}$ cm, the gravitational potential at every point is numerically close to the quantity c^2. This clarifies why one part of the deflection is always given by an invariant factor, where, surprisingly, the velocity of light appears even while considering the gravitational deflection of other material particles or waves.

It is important to stress why this derivation gives a result identical to the full general relativistic result in which no wave aspect of light or particle is directly used. The non-Newtonian part of the deflection in general relativity comes from consideration of the curvature of the space-time in which the deflection occurs close to the massive body. The spatial coordinate system near the massive object is general relativistically distorted with respect to the asymptotic coordinate system in which the measurements are made. The gravitational factor that contributes to the additional deflection is same as the factor that is responsible for the gravitational redshift of a wave in the gravitational field. Therefore, whether we use an extended entity like a wave to include the variation of the time dilation factors in the gravitational field, or use the curvature of space-time caused by the mass, the results are the same. In other words, the plane wavefront is like a coordinate grid in the direction perpendicular to the wave vector, and the gravitational redshift and resulting bending are exactly like the curvature of any imaginary coordinate grid. What is important is to note that such a contribution is fundamentally different from the deflection obtained by applying the equivalence principle to a ray or a particle. This need not mean that general relativistic gravity is inconsistent without the quantum nature and wave aspect of particles to which the theory applies. However, one may argue that point particles without a quantum mechanical wave aspect are not fully consistent with general relativity in a local analysis, since point particles are singularities of the field. This issue needs to be studied further. What general relativity needs is the relativistic changes of scales and clocks as the position is changed in the gravitational potential. These effects can be incorporated either by correctly taking into account of the transformation between coordinates as usually done in the general relativistic derivation of the deflection light, or by considering physical effects on spatially extended objects, like waves. Both are equivalent as far as the final result for deflection is concerned, though there are important differences in the conceptual and physical aspects of the two approaches. The discussion presented here using explicitly the physical effects on waves in a gravitational field is by far the simplest, and most transparent.

1. The English translation of the 1911 paper of Einstein in *Ann. Phys.* is reproduced in '*The Principle of Relativity*', Collection of Original Memoirs, Dover Publications Inc, New York, 1923.
2. See any textbook on general relativity and gravitation for the general relativistic derivation of the light deflection formula, and for the derivations of the redshift formula.
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ACKNOWLEDGEMENTS. I thank Prof. Anna Nobili for several discussions on the historical and the conceptual aspects of the results presented here. I thank Saptarishi Chaudhury, Sanjukta Roy and D. Suresh for discussions on the gravitational deflection of sound waves.

Received 25 October 2004; revised accepted 14 February 2005

Preparation and characterization of magnesium ion conducting glass–polymer composite films

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The flexible glass–polymer composite film electrolytes of high magnesium ion conducting oxysulfide glass and the comb-shaped poly(oxyethylene) polymer (TEC-19) were prepared. The conductivity of the composite film with 2% (v/v) TEC-19 doped with $Mg(ClO_4)_2$ was 1.2×10^{-3} S cm^{-1} at 100°C and 3.5×10^{-5} S cm^{-1} at 30°C. The composite exhibited a 4V stable potential window versus Mg^{2+}/Mg. The flexible glass–polymer composite films prepared in this way are promising solid electrolytes for solid magnesium secondary batteries. The electrical conductivity for pelletized composite film was measured in a dry Ar atmosphere by the ac impedance method in the temperature range 25–250°C and the frequency range 1 Hz–10 MHz. Temperature dependence of electrical conductivities of glass–polymer composites has been observed.

IONIC conducting polymers[1,2] have been of considerable interest because of their commercial application as batteries, signal processing devices, vacuum tubes, semiconductors, fibre-optics, charge-transfer complexes, etc. Many workers[3] have reported synthesis and other characteristic data on the

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