Classical limit for quantum mechanical energy eigenfunctions

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The classical limit problem is discussed for the quantum mechanical energy eigenfunctions using the Wentzel–Kramers–Brillouin approximation, free from the problem at the classical turning points. A proper perspective of the whole issue is sought to appreciate the significance of the discussion. It is observed that for bound states in arbitrary potential, appropriate limiting condition is definable in terms of a dimensionless classical limit parameter leading smoothly to all observable classical results. Most important results are the emergence of classical phase space, keeping the observable distribution functions non-zero only within the so-called classical region at the limit point and resolution of some well-known paradoxes.

THE quantum–classical relationship is studied in disparate ways with varied motivations. Semiclassical methods, built on classical trajectories, have their roots in the old quantum theory. The knowledge of the classical behaviour (regular and 'softly' or 'harshly' chaotic) is used to draw conclusions about the quantum mechanics (QM) of the system starting from the Green's function or the propagator and then reducing its trace to a sum over all periodic orbits[1]. A general scheme is thus devised to handle the extreme cases of classical dynamics – both integrable and ergodic. With the derivation of 'trace formula', the semiclassical theory hopes to overcome the challenge posed by classical chaos and casts light on the correspondence principle[2]. Berry has clearly identified the Holy Grail of the semiclassical theory as the quantization of classical chaos. The main emphasis of the above approach is to formulate a tractable and reliable theory of the so-called 'mesoscopic' systems and not as such to address the problem of classical limit and the emergence of classical behaviour from QM. According to Gutzwiller[1], 'On the whole, the calculational results, while acceptable or even good, cannot be expected to be correct to the last decimal; but they are at least understandable intuitively, rather than being the result of a monstrous numerical calculation'. Exact quantum mechanical calculations become exceedingly difficult near the classical regime and a sort of 'classical version of QM' (as Gutzwiller himself branded his theory) comes handy to probe many unexplored issues.

Sen *et al.*[3] have shown that Ehrenfest's theorem, in its generalized form, reveals some interesting exact relations between classical mechanics and QM. An idea of approximate particle tracks emerges from closed relations among few quantum mechanical expectation values in certain systems according to Ehrenfest's theorem. The analysis clearly explains why in all early experiments on electrons, protons, etc. classical mechanics has been successfully used and even in precise experiments as in mass spectrometers or β-ray spectrometers, no deviation from classical laws was detected. We would like to point out a popular viewpoint among many physicists[4,5] that for all practical purposes, classical behaviour of macroscopic bodies can be easily traced in terms of narrow packet representation and Ehrenfest's theorem. For macroscopic masses, spreading time turns out to be of the order of cosmological period and may be neglected. The problem of superposition of two packets also cannot exist because there is continuous observational interaction with the surroundings and the classical particles always remain in a collapsed narrow state. Ballentine *et al.*[6] have criticized this view because the classical equation of motion does not exactly agree with Ehrenfest's equation. But a more serious objection has been raised by Sen and Sengupta[7]. For classical particles it is always possible to know both position and momentum with finite errors, i.e. both position and momentum density functions must be compact. This is impossible in the ψ-function representation.

Based on trajectories, two independent formulations differing in conceptual structure but mathematically equivalent with standard QM have been developed. These are Feynman's path integral approach and Bohm's causal QM respectively, which definitely enriched our understanding of the problem. In Feynman's approach, a quantum mechanical system samples all possible paths having different weight factors (nonanalytic paths too!) and the motion is described as weighted sum over all paths. In the appropriate classical limit, the amplitude concentrates itself about the stationary action (classical) path. Since this method is extremely difficult to apply to solve simple energy eigenvalue problems, it is not a practical proposition to use this approach for discussing the classical limit problem

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for eigenfunctions of energy. The Bohmian approach, on the other hand, transfers the inescapable nonlocality of quantum description from the state function to the so-called quantum potential (Q) and accommodates the so-called particle ontology by discarding completeness claim of the wave function description of an individual system. The theory, however, presents what is called 'unobserved reality' instead of the observed one and it is argued that a state-dependent condition $Q \to 0$ (vanishing quantum potential), encapsulates within a single universal criterion for characterizing and achieving the classical domain as a mathematical limit of QM. An additional condition is, therefore, necessary to prove that in the limit $Q \to 0$, the motion will turn out to be the observed one. But even if $Q = 0$, the Bohmian tracks are not classical in all respects because these tracks are completely determined by the initial position of the particle, while for classical tracks we need to know both initial position and velocity. To emphasize the inherent difficulty of this approach, we consider two examples. For a free particle wave function $\psi = \exp(i\,\vec{p}.\vec{r}/\hbar)$, $Q = 0$. The only meaning of this result is that the unobserved motion of the particles will obey classical laws. If we try to introduce classical ontology here, there will be difficulties. Our second example is that of a Gaussian packet for which $Q \neq 0$, but still the propagation of position and momentum density functions are found to obey classical laws[8]. Thus even with non-vanishing quantum potential, we may have classical behaviour for quantum mechanical systems.

Another line of argument maintains that whatever limiting procedure is adopted, the macrorealistic description cannot emerge from standard QM unless additional ingredients are introduced. Serious attempts have been made to understand the appearance of classical regime in terms of environment-induced decoherence models[9]. However, all aspects of classical realities are yet to be explained consistently in the decoherence scheme using standard QM. The theories developed by Ghirardi *et al.*[10], Diosi[11] and Joos and Zeh[12] modify the Schrödinger equation, introducing nonlinearity that leads to spontaneous collapse and localization in space occurring at random times. The decoherence models (environment-induced or dynamical), lack the 'particle ontology' of the classical systems because they adhere to the standard assumption that the wave function description provides complete specification of the state of an individual system. Moreover, Ballentine[13] has shown that some of these (spontaneous collapse models) theories violate the energy conservation principle and are incompatible with the existence of stationary states.

Yaffe[14] has proposed a novel approach based on the observation that increasing the number of degrees of freedom (N) simplifies the analysis in many quantum mechanical problems. The author asserts that in the classical limit all quantum interference effects must disappear. The prescription for obtaining the classical limit of quantum theory involves (i) construction of general coherent states[15,16] by applying each of the elements of a Lie group of unitary operators (having the same dynamical symmetry group of the physical system under consideration), that acts irreducibly on the Hilbert space of the theory, to some initial state $|0>$, and (ii) the demonstration that any coherent superposition of these states becomes in the limit indistinguishable from an incoherent mixture of the same. The irreduciblity of the group apparently implies the completeness of the basis. However, rigorous mathematical demonstration of the completeness of the coherent state basis functions with the required classical limit property in the large N limit is yet to be achieved.

Coherent states, first discovered by Schrödinger[17] and later extended by others, seem to offer a simple approach to the classical limit problem. In a harmonic oscillator potential both the position and momentum probability densities $\rho(x)$ and $\bar{\rho}(p)$ execute exact classical motions and in the limit $\hbar \to 0$, both density functions become localized and it appears that we have arrived at the classical particle description from quantum principles. Hepp[4] tried to extend this result to arbitrary potential functions. But the argument is fallacious. Let us take the Schrödinger coherent state in a harmonic oscillator potential:

$$\psi(x,t) = N \exp\left[-\frac{\beta^2 (x - x_{\mathrm{cl}})^2}{2} - \frac{iE_0 t}{\hbar} + \frac{ip_{\mathrm{cl}}}{\hbar}\left(x - \frac{x_{\mathrm{cl}}}{2}\right)\right],$$

where $\beta^2 = \mu\omega/\hbar$, $x_{\mathrm{cl}}(t) = A\cos\omega t$, $p_{\mathrm{cl}}(t) = \mu\dot{x}$ and $E_0 = \hbar\omega/2$.

To recover the classical picture of an oscillating particle, it is necessary that both $\rho(x)$ and $\bar{\rho}(p)$ become localized to a point in a suitable limit. In terms of the dispersions $\sigma_x^2 = <x^2> - <x>^2 = \hbar/(2\sqrt{\mu\kappa})$ and $\sigma_p^2 = <p^2> - <p>^2 = \hbar^2/(4\sigma_x^2)$, ($\kappa$ being the spring constant), a dimensionless classical limit parameter

$$\alpha = \frac{\sqrt{2\mu < E >}\sigma_x}{\hbar} = \sqrt{\frac{\sqrt{\mu} < E >}{\hbar\sqrt{\kappa}}}; < E >= \frac{1}{2}(\mu\omega^2 A^2 + \hbar\omega)$$

can be defined. If we go to the limit $\alpha \to \infty$ along the path $\hbar \to 0$, keeping μ, $< E >$ and κ constant, both $\rho(x)$ and $\bar{\rho}(p)$ become localized to a point. But this is unrealistic because (i) in nature, $\hbar$ is an universal constant and (ii) there are examples where quantum mechanically predicted results[18] are independent of $\hbar$. The correct limiting path might be $\mu \to \infty$, keeping $\hbar$, $< E >$ and κ constant. But in this procedure though $\rho(x)$ becomes localized, $\bar{\rho}(p)$ spreads without limit and the expected point particle picture cannot emerge. This result gives a convincing proof that a ψ-function representation (narrow packet) per se cannot provide classical-like description of single particle dynamics.

Since the predictions of quantum theory are in the form of probabilities, the emergence of classical behaviour from

quantum mechanical energy eigenvalue problems is usually studied by comparing the position and momentum distributions for a classical ensemble with those obtained from QM in the limiting situation. The phase-space formulation of QM offers a framework in which quantum phenomena can be described using as much classical language as allowed. The main tool of the phase-space formulation is the phase-space distribution function defined in terms of the expectation value of a quantum mechanical operator corresponding to a function of classical conjugate variables q, p. Wigner distribution function (WDF) determined from the quantum mechanical ψ-function describes correctly the position and momentum density functions, but for other functions such as energy, it is the expectation values which are given correctly. Actually, it yields correct quantum mechanical expectation values for all Hermitian-ordered operators calculated through phase-space integration (the Weyl rule of association). Also (i) it satisfies the Liouville's equation only approximately and (ii) it is well known that it is not always positive definite. Fulfilment of the last two conditions is necessary to ensure that in the classical limit, WDF can be interpreted as a physically meaningful classical phase-space distribution function. In fact, Cohen[19] has clearly demonstrated that no choice of distribution function can furnish an exact reformulation of QM in terms of a phase-space density function. However, the quantum–classical correspondence demands that exact phase space formulation of QM should be possible, at least in the classical limit. Home and Sengupta (unpublished report) have observed that in the case of analytic potential functions and for dynamical variables which are analytic functions of the Cartesian position and momentum coordinates, WDF satisfies an exact phase-space formulation of non-relativistic QM in the limit $\hbar \to 0$. However, this very limiting procedure and the underlying expansion in terms of $\hbar$ remain questionable.

The conventional approach in the classical limit problem, on the other hand, focuses our attention to specific problems and to search for the limiting conditions at which the quantum mechanical results agree with those obtained from classical mechanics, ignoring the bigger question of retrieving a whole domain of classical observable results from QM. In such cases, we like to emphasize that the meaning of the classical limit is not always clear and in different cases the meaning may be different. Various problems like scattering of a particle in potential fields[20,21], wave-packet propagation in free and potential fields – calculation of travel time, the study of barrier penetration and atomic collision dynamics[22,23], classical limit of quantum radiation fields[24,25] and the limiting behaviour of quantum mechanical stationary states[26,27] have been considered by different authors. These results are interesting and useful, but do not clarify the extent of the conceptual framework of classical mechanics recovered from the quantum principles. For example, in scattering problems the classical expression for differential cross-section can be obtained in a limiting way, but the classical picture of particle tracks deviated by interaction with the scatterer, remains unrecovered. We insist that in the classical domain this picture implied in the classical calculational method is a valid one, because the tracks are experimentally verifiable.

In a previous work[21], we have developed a method of obtaining the classical limit for scattering of particles by a potential field in terms of a dimensionless limit parameter. In the present work we discuss a general method to get the classical limit of quantum mechanical energy eigenstates. We use the WKB method[28,29] for asymptotic solutions which circumvent the problem at the classical turning points. After Pauli[30], it is widely believed that in the limit $\hbar \to 0$, WKB asymptotic solution of Schrödinger equation leads to the Hamilton–Jacobi (H–J) equation of classical mechanics. This is a myth similar to the belief that Ehrenfest's theorem leads to Newton's equation of classical mechanics. The latter has largely been removed by the recent works of Sen *et al.*[3] and Ballentine *et al.*[6]. But little has been written on the inadequacy of the usual treatment of WKB approximation in the classical limit problem[31]. Formulations by Pauli[30] and by Bohm and Hiley[32] arrive at a classical look-alike 'quantum H–J equation', replacing ∇S (or ∇S_0 to the first order in Pauli's case) by the momentum. This quantity, termed as 'unobserved momentum' of the system by Bohm, is identically equal to zero in the case of eigenfunctions in one dimension. In the context of the classical limit problem this resemblance is, therefore, thoroughly illusory. As Holland[18] has rightly pointed out that 'even if one contrives to obtain the classical H–J equation in some limit, we are not justified in identifying this with the H–J equation describing the propagation of an S-function associated with an ensemble of precisely defined trajectories whose law of motion is given by $\dot{x} = \nabla S/\mu$ or in treating the probability of being as a limit of the probability of finding. . . . In order to smoothly connect quantum and classical dynamics in the usual approach, the law $\dot{x} = \nabla S/\mu$ and the notion of a precise initial position x_0 of a material object are slipped in as additional postulates'.

Textbooks[33,34] and pedagogical articles[27,35] often establish the quantum–classical relation by showing that the two distribution functions approach each other, in a locally averaged sense, in the usual correspondence principle limit (Bohr) of large quantum number n. First, we note that no physically meaningful mechanism other than imprecise detecting process can be associated with the local averaging procedure referred to as 'coarse-graining'. Also, the limiting procedures $n \to \infty$ or $\hbar \to 0$ are not always defined in physically meaningful ways. Particularly important is to identify a dimensionless limit parameter and to mention precisely which variables are kept constant as the parameter approaches the limit, so that the limiting path is clearly defined. The problem may be highlighted

by the example discussed by Cabrera and Kiwi[36] that even if individual energy eigenfunctions lead to the required classical results in the large quantum number limit, simple linear combination of three one-dimensional harmonic oscillator eigenstates gives results at variance with classical physics. Earlier, Home and Sengupta[37] constructed a paradoxical example with superposed eigenfunctions of one-dimensional Coulomb potential, which leads to inconsistency with classical mechanics in the limit $n \to \infty$. The present analysis shows that these incongruities are removed with the adoption of proper limiting procedure. We will discuss the resolution of the paradoxical examples in detail later in the article.

The limit $\hbar \to 0$ is also not physically meaningful unless compared to which relevant parameter of the system $\hbar$ is small[18,21,31]. In fact, all limiting processes must be defined in terms of a dimensionless limit parameter. Another aspect almost wholly ignored is the lack of realization that it is not enough to say that the dimensionless parameter tends to zero/infinity in the limit. The statement for the limiting procedure is incomplete unless one states which of the dynamical variables are kept constant as the parameter approaches the limit. Otherwise significant variables like energy, angular momentum, etc. will become unphysical at the limit point. For example, with $|E| \propto (n\hbar)^{-2}$ for hydrogen atom, does the limiting process $n \to \infty$ or $\hbar \to 0$ imply that in the limit the energy of the system tends to zero or infinity?

Almost all textbooks on QM mention that classical results are achievable through WKB approximation without demonstrating which particular results are achieved. The first difficulty in the WKB approximation is with the limiting process. Our assertion is that (i) we have to find a suitable dimensionless parameter and (ii) completely specify the path along which the limit is reached by indicating which dynamical quantities are to be maintained constant. In this article we confine our attention to bound-state eigenfunctions only. It is shown here that a dimensionless limit parameter α can be defined in terms of the separation between the two turning points. The transcribed WKB equation in terms of α, renders asymptotic solutions true in the limit $\alpha \to \infty$. The exact path along which the limit is reached is determined by keeping the energy of the system constant. In the limiting situation, only the observational predictions of the theory, i.e. the distribution functions (both position and momentum) remain finite and meaningful. To our knowledge, there exists no treatment of the WKB approximation along the above lines. It is also observed that the conventional approach to the classical limit problem starting from the Schrödinger wave equation, provides a physically transparent and comprehensive framework for the understanding of the emergence of classical mechanics in a limiting way. For scattering states, the WKB method is not helpful and the problem will be discussed elsewhere.

In QM, the coordinate-space and momentum-space distributions are correlated via Fourier transformation relation of the corresponding wave functions. The method of stationary phase, using WKB wave function, gives the stationary phase point x_0 as a function of p from which comes the significant contribution to the Fourier integral. In the limiting situation, the method finally yields the classical momentum distribution as well as the phase-space distribution, even though the phase-space distributions have no counterpart in standard quantum theory. This is due to the fact that in the classical limit, the method of stationary phase gives rise to a unique correlation between position and momentum, which in a way implies snapping-off the Fourier coupling of the coordinate and momentum spaces and consequently the breakdown of (x, p) uncertainty relation. In the development of the semiclassical approximations in QM, both Gutzwiller[1] and Maslov and Fedoriuk[38] have used the method of stationary phase. In the present study, however, this method assumes greater significance as it leads to the emergence of the limiting phase-space description from a ψ-function.

WKB solution for bound states

The Schrödinger equation for one-dimensional eigenvalue problem may be written in the form:

$$\alpha^{-2} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = Q(x)\psi \,, \tag{1}$$

where $\alpha^2 = 2\mu|E|\alpha^2/\hbar^2$, and $Q(x) = -a^{-2}[1 - V(x)/E]$, E is an eigenvalue of energy for a bound state, $V(x)$ is the potential energy function and μ is the mass of the particle. We assume $V(x)$ to be analytic and such that there are two turning points ($Q = 0$) at $x = A$, B ($B > A$) and $a = B - A$. Between the two turning points, the dominant terms of WKB solution (terms of the order of α^{-1} are neglected)[28],

$$\psi(x) \sim [-Q(x)]^{-1/4} \sin\left[\alpha \int_x^B \sqrt{-Q(t)}\,\mathrm{d}t + \frac{}{4}\right]. \tag{2}$$

In the region beyond B ($x > B$) the solution is

$$\psi(x) \sim [Q(x)]^{-1/4} \exp\left[-\alpha \int_B^x \sqrt{Q(t)}\,\mathrm{d}t\right], \tag{3}$$

and for $x < A$,

$$\psi(x) \sim [Q(x)]^{1/4} \exp\left[-\alpha \int_x^A \sqrt{Q(t)}\,\mathrm{d}t\right]. \tag{4}$$

For a number of problems (e.g. linear harmonic oscillator, particle in a homogeneous field etc.), Flügge[29] has shown that the WKB method correctly reproduces quantum mechanical asymptotic solutions.

Continuity requirement of the eqs (2)–(4) leads to the general WKB eigenvalue formula:

$$\alpha\int_A^B\sqrt{-Q(t)}\,\mathrm{d}t=\left(n+\frac{1}{2}\right)\quad, \tag{5}$$

where $n = 0,1,2, \ldots$. The equation is asymptotically true, i.e. terms of the order α^{-1} on the right hand side are neglected. An inspection of eq. (5) reveals that in general, the eigenvalue E must be a function of $n\hbar/\sqrt{\mu}$. This result may be easily checked for the special cases of harmonic oscillator or hydrogen atom problem. We define a dimensionless quantity:

$$\alpha=\frac{\sqrt{2\mu\,|\,E\,|}a}{\hbar} \tag{6}$$

as the classical limit parameter such that the classical regime is recovered in the limit $\alpha\to\infty$, when the limit is taken keeping E and other relevant dynamical quantities constant. Since E is a function of $n\hbar/\sqrt{\mu}$, this can be achieved (i) for fixed μ letting $\hbar\to 0$, $n\to\infty$ such that $n\hbar$ is constant and (ii) for the fixed observed value of $\hbar$, taking $n\to\infty$ and $\mu\to\infty$, keeping $n/\sqrt{\mu}$ constant. The implications and significance of the limiting procedures will be discussed presently and we argue for a paradigm shift on some interpretational aspects of the classical limit problem. We see from eqs. (3) and (4) that outside the region AB, both $\psi(x)$ and $|\psi(x)|^2$ tend to zero as $\alpha\to\infty$, in agreement with the classical result. Between the turning points in the region AB, the probability distribution function in the limit is

$$\rho(x)\mathrm{d}x\propto|\psi(x)|^2\,\mathrm{d}x\sim\frac{\mathrm{d}x}{\sqrt{-Q(x)}}. \tag{7}$$

Because of the rapid oscillations, the squared sine factor in $|\psi(x)|^2$ (from eq. (2)) approaches 1/2 in the limit $\alpha\to\infty$ according to Riemann–Lebesgue lemma. For the conventional limiting process, when we confine to a state of high energy in the asymptotic region, a coarse graining process is essential because the oscillations of the density function have a finite spacing. In our limiting procedure ($\alpha\to\infty$, keeping the energy constant), the oscillations of the density function become infinitely close and we can use the rigorous mathematical procedure of the Riemann–Lebesgue lemma (for squared sine function). But the infinitely rapid oscillation of the wave function robs its effectiveness as a mathematical aid to calculation. However, starting from the usual standard form of the Riemann–Lebesgue lemma, which states that the integral of the product of a rapidly oscillating function and a smooth function goes to zero, if we take the second function as constant, it implies that the integral of a rapidly oscillating function will go to zero. As the classical limit is approached, for the infinitely close rapid oscillations of the function, the average value of the integral over arbitrary small intervals decreases monotonically and becomes zero at the limit point, making the function itself equal to zero. It implies the significant result that in the limiting situation, all the physical quantities like position density function, etc. remain finite with a gradual weakening of the wave function ($\psi\to 0$).

In many quantum mechanical calculations one has to determine the limiting value of oscillating functions like e^{ikr}, $r\to\infty$ (such as Born's approximation formula for Rutherford scattering). The limit is generally taken as zero, but the logic is usually not mentioned. Consider the measurement of a physical quantity represented by, say, the function $\sin(\alpha x)$. The instrument interacts with the physical system not exactly at a point x but over a small length Δx and the response is really the average value of the function over Δx around x. 'The value of $\sin(\alpha x)$ at x' really means the quantity

$$<\sin(\alpha x)>=(\Delta x)^{-1}\int_{x-\Delta x}^{x+\Delta x}\sin(\alpha x)\mathrm{d}x$$

$$=\frac{2\sin(\alpha x)\sin(\alpha\Delta x)}{(\alpha\Delta x)}\to 0\ \text{ as }\alpha\to\infty$$

This limit is independent of the smallness of Δx. The difference between this limiting procedure and coarse graining is clear. If α is kept fixed at a constant value, then the period of oscillation in $\sin(\alpha x)$ is $\propto\alpha^{-1}$. If Δx is small enough such that $\alpha\Delta x \ll 1$, then $<\sin(\alpha x)>$ remains oscillating with respect to x. If $\alpha\Delta x \gg 1$, i.e. Δx is much greater than the spacing of oscillations, then only $<\sin(\alpha x)>$ will give an almost smooth function. This procedure is coarse-graining. Thus coarse-graining implies a constant α, whereas Riemann–Lebesgue lemma implies a limiting process with $\alpha\to\infty$.

The Fourier transform of $\psi(x)$ will give us the momentum probability density $\bar{\rho}(p)=|\Phi(p)|^2$, where

$$\div(p)=\frac{1}{\sqrt{2\ \hbar}}\int\psi(x)\exp[-ipx/\hbar]\mathrm{d}x\,. \tag{8}$$

Near the limiting situation, $\psi(x)$ becomes vanishingly small beyond the turning points; we can therefore confine the integral eq. (8) between A and B and substitute for $\psi(x)$ from eq. (2). Writing the sine function in terms of exponential functions, the RHS of eq. (8) breaks up into two terms as $\Phi(p)=\Phi_+(p)+\Phi_-(p)$, where

$$\div_{\pm}(p)=\frac{c_{\pm}}{2i}\int\left[\{-Q(x)\}^{-1/4}\exp\pm i\{\alpha\int_x^B\sqrt{-Q(t)}\mathrm{d}t+\frac{\pi}{4}\mp\frac{px}{\hbar}\}\right]\mathrm{d}x\,. \tag{9}$$

We use the method of stationary phase[21,39] to evaluate the integral. In the neighbourhood of the classical limit, most of the contribution to the integral comes for

values of x near the stationary phase point x_0 defined by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha\int_x^B\sqrt{-Q(t)}\mathrm{d}t+\frac{}{4}\mp\frac{px}{\hbar}\right]=0$$

or

$$\frac{p^2}{2\mu}=E-V(x)\,. \tag{10}$$

The solution of eq. (10) gives the stationary phase point x_0 as a function of p. It implies that significant contribution to the integral of eq. (9) comes from the neighbourhood of x_0. In other words, we may also say that x_0 and p become correlated by the classical relation given in eq. (10). The phase term within the exponential function in eq. (9) may be expanded about x_0 retaining terms up to δ^2, where $x = x_0 + \delta$. We then ultimately have

$$\theta=\alpha\int_{x_0}^B\sqrt{-Q(t)}\mathrm{d}t\pm\frac{1}{\hbar}\left[\frac{1}{2}(\frac{\mathrm{d}p}{\mathrm{d}x})_{x_0}\delta^2\mp px_0\right], \tag{11}$$

and

$$\div_{\pm}(p)\propto\exp\pm i\left[\alpha\int_{x_0}^B\sqrt{-Q(t)}\mathrm{d}t\mp\frac{px_0}{\hbar}\right]\int\frac{\exp\pm i(\gamma_0\delta)^2}{[-Q(x_0+\delta)]^{1/4}}\mathrm{d}\delta\,, \tag{12}$$

where

$$\gamma_0^2=\frac{1}{2\hbar}(\frac{\mathrm{d}p}{\mathrm{d}x})x_0\,.$$

Since most of the contribution to the integral comes from the region around $x = x_0$, we can take the denominator in the integrand outside. Writing this term as proportional to $\sqrt{p(x_0)}$ and putting $\gamma_0\delta = z$, we get

$$_{\pm}(p)\propto\frac{\exp\pm i\alpha\left[\int_{x_0}^B\sqrt{-Q(t)}\mathrm{d}t\mp\frac{px_0}{\hbar}\right]}{\gamma_0\sqrt{p(x_0)}}\int\exp\pm iz^2\mathrm{d}z\,. \tag{13}$$

Again, since γ_0 is proportional to α, it goes to ∞ in the classical limit and the entire contribution to the integral comes essentially from the point $\delta = 0$; we can take the limits of the integral from $-\infty$ to ∞. The modulus of the integral is $\sqrt{}$ and the momentum distribution, therefore, may be written as

$$\bar{\rho}(p)\mathrm{d}p=|\div(p)|^2\,\mathrm{d}p\propto\frac{\mathrm{d}p}{p\frac{\mathrm{d}p}{\mathrm{d}x}}\,.$$

Now if we interpret eq. (10) as defining the momentum at x of a particle with energy E, this is identical to the classical value of $\bar{\rho}(p)$.

Finally, let us summarize all the limiting properties of the quantum ensemble with members having the same energy E. The position and the momentum densities within AB are given by $\rho(x)\propto 1/\sqrt{E-V(x)}\sim 1/p$ and $\bar{\rho}(p)\propto 1/(p\frac{\mathrm{d}p}{\mathrm{d}x})$, respectively, where p and x are related by eq. (10). Also $\frac{\partial\rho}{\partial t}=0$ and outside AB, both $\rho(x)$ and $\bar{\rho}(p)$ become zero. All these are quantum results. If we now insist that particles found between x and $x + \mathrm{d}x$ have momentum between p and $p + \mathrm{d}p$ where p is given by eq. (10), then all the above properties belong to a classical ensemble for which the initial $\rho(x)\sim 1/\sqrt{E-V(x)}$. But for a classical periodic system $\rho(x)$ is proportional to the time spent in the region between x and $x + \mathrm{d}x$, i.e. $\rho(x)\propto\dot{x}^{-1}$. Equivalence of quantum and classical distribution functions at the limit point, therefore, implies the classical relation $p/\mu = \dot{x}$. We see that the same physical facts $\rho(x)$, $p(x)$, etc. can now be interpreted by an alternative conceptual structure where position of a particle at a time t attains ontological significance and velocity of a particle is a fundamental observable quantity. The self-contained classical mechanics is indispensable for QM to ensure a direct contact with the observable world (through the postulates of measurement theory). A complete agreement of both the position and momentum distribution functions at the classical limit point is, therefore, absolutely necessary for the theory reduction to recover the entire observational results. This is exactly what is established here for quantum mechanical energy eigenfunctions. We have also found that at every x, there are two possible values of momentum given by eq. (10). This result indicates a breakdown of the uncertainty principle at the limit point. This removes the major hurdle in introducing the concept of particle tracks.

The constancy of $\rho(x)$ follows from the classical continuity equation:

$$\frac{\partial\rho(x)}{\partial t}=-\frac{\partial[\rho(x)\dot{x}]}{\partial x}=0\,. \tag{14}$$

Thus the limiting quantum results exactly agree with those of a classical ensemble moving according to the classical laws. The phase space of this ensemble is given by

$$W(x,p)=\rho(x)\,\delta\,(p-p(x)). \tag{15}$$

We have remarked earlier that the limiting process $\alpha\to\infty$, keeping E constant, can be reached in two ways. In the first instance we keep μ constant and take $\hbar\to 0$, $n\to\infty$ such that $n\hbar$ remains constant. This may be regarded as a theoretical limiting process because we are changing the value of the universal constant $\hbar$ and in a way, gradually eliminating all quantum effects from nature at the limit point. The second procedure is to keep $\hbar$ constant and vary $(\mu, n)\to\infty$ such that $n/\sqrt{\mu}$ remains constant. This means that for heavy particles, quantum effects will gradually decrease.

It is clear that the discreteness of energy which depends on $\hbar/\sqrt{\mu}$ also disappears in the limit $\alpha\to\infty$, keeping $\hbar$ and E constant. By a reasonable choice of accuracy we

can fix α at a sufficiently large value say $\alpha_0 \gg 1$, so that for a given energy E, we can find a critical value of mass μ_0, such that for $\mu > \mu_0$, quantum effects will be negligible. Thus nature is divided into a quantum domain of low mass and a classical domain of large mass.

Using quantum mechanical asymptotic solutions and taking conventional local averages (coarse-graining) for the density functions, Robinett[27] has compared the quantum and classical probability distributions for both position and momentum for several model one-dimensional systems. We have arrived here at the general result with precise definitions of the limit parameter and of the limiting path. Moreover, the present method recovers the phase-space at the limit point and the following discussion will show how it resolves paradoxical examples of classical limit of superposed states.

For two energy eigenvalues E_1 and E_2, we write for the superposed state (from eq. (2)):

$$\psi(x) \sim \psi_1(x) + \psi_2(x) \sim [\{-Q_1(x)\}^{-1/4} \sin \alpha_1\theta_1(x) + \{-Q_2(x)\}^{-1/4} \sin \alpha_2\theta_2(x)]. \quad (16)$$

Then the interference term in the position density function:

$$(Q_1(x)Q_2(x))^{-1/4} \{\sin \alpha_1\theta_1(x) \sin \alpha_2\theta_2(x)\}$$

will vanish as both $\sin \alpha_1\theta_1(x)$ and $\sin \alpha_2\theta_2(x)$ independently approach zero, when the limit $\alpha \to \infty$ is taken keeping the energy values constant. Thus in the limit the original pure quantum state will be reduced to a mixed state of different energies. Particles of different energies at the same position x will have different momenta. This resolves the paradox posed by Cabrera–Kiwi and Home–Sengupta examples and implies the fact that the large quantum number is not a proper classical limit criterion. According to the present analysis, classical limit is approached in the large α limit whence the cross or interference terms disappear with the individual wave functions approach to zero at the limit point. The change of a pure quantum state to a mixture of various energy states in the classical limit is similar to a decoherence effect. We note here that the passage to classical limit is entirely different from the unitary time evolution process implied by Schrödinger equation, which cannot reduce a pure state to a mixed state. In this case the decoherence effect is induced by a change of the limit parameter and not by a time development.

Discussion

All previous attempts to get the classical limit for eigenstates of energy are confined to specific potential functions. In the present analysis it has been possible to develop the classical limit for arbitrary analytic potential functions. The three characteristic features of the present approach are: (i) a dimensionless limit parameter is defined and the asymptotic WKB solutions are expressed in terms of this parameter, (ii) a precise limiting path is defined using the condition that the energy E of the system will remain constant all through the limiting process and (iii) the method of stationary phase is invoked to calculate the Fourier transform. This procedure leads to the emergence of a correlation between p and x at the limit point and finally yields all the observable classical results from the quantum description of bound systems in the appropriate limit.

In the classical regime, $\rho(x)$ becomes entirely confined between the two turning points and as E is kept constant, momentum is also definitely confined within a finite interval. Such a state is no longer describable by a ψ-function[7]. A spatially narrow packet is bound to be inordinately extended in momentum space and cannot represent a classical particle having compact probability density functions in both coordinate and momentum spaces. Description of a realistic classical particle demands the breakdown of Fourier correlation inherent in wave function representation. In the present analysis, the correlation between x and p obtained in eq. (10) can be interpreted that a particle at position x has the momentum $p(x)$. However, these results alone do not lead to classical mechanics, since complete equivalence of quantum and classical distribution functions at the limit point requires the classical relation $p/\mu = \dot{x}$, where velocity of a particle ($\dot{x}$) is a fundamental observable quantity. In QM, velocity is undefined and we have an abrupt qualitative change here. It should be emphasized that the transition to classical mechanics is achieved by introducing the interpretational postulate that the momentum $p(x)$ is related to classical velocity $\dot{x}$ through the relation $p/\mu = \dot{x}$. Equation (10) leads to Newton's equation of motion and all properties of the quantum system are consistently interpretable as those of a classical ensemble defined by the phase space density function given in eq. (15).

This is somewhat similar to the emergence of new concepts and qualitative features when the general theory of relativity (GTR) is reduced to Newton's gravitational theory in the limit of weak field and small velocity. By changing the interpretation of 'g_{44}' as a gravitational potential function, the concept of 'force' emerges along with the linearity of the basic dynamical equation. This leads to the uniqueness theorem for the solutions and to the Newtonian determinism, all of which are alien to the nonlinear GTR. All these new features are experimentally well verified in this domain. It is obvious that a change in conceptual paradigm cannot be deduced mathematically. It requires the introduction of interpretational postulates. When two theories use essentially the same conceptual framework, a purely mathematical reduction of the general theory to the restricted one is possible. Even in the case of special theory of relativity, Bacry and Levy-Leblond[40] have shown that the low velocity limit of Lorentz

transformations for space-like intervals ($(x_2 - x_1)/c(t_2 - t_1) \gg 1$) shows non-Galilean features as the time-ordering of two such independent events may be reversed for some observers. The concept of absolute time in classical mechanics is valid only for a finite world!

In the present study our analysis shows a gradual weakening of the wave function to zero as the classical limit point is approached, indicating that quantum calculations become increasingly difficult to work out and classical calculations give more and more accurate predictions. We have extended the present method (to be discussed elsewhere) to two- and three-dimensional cases and considered systems under non-analytic potentials such as discontinuous potential barriers where the WKB method is not applicable. In all the problems discussed, it appears that QM is valid in the classical regime in a limiting way. However, some of the hallmark notions of QM (e.g. wave function description, quantum uncertainty relations, etc.) cease to be operative in the classical domain.

We have discussed the inadequacies of quantum mechanical wave function description for a macroscopic system (in a subsequently communicated work). The significance of the unphysical, infinitely rapid oscillations of the wave function near the classical regime has been critically examined here, where the inadequacy of quantum mechanical wave function for a complete description of macroscopic system is discussed. When we separate out the centre of mass motion from the many-particle Schrödinger equation describing a macroscopic body, it represents an equation for a massive point particle. Our analysis of classical limit problem shows that in the case of heavy mass, classical description emerges with the disappearance of all the nonclassical features of QM. It is further argued that appearance of particle track is essentially a classical phenomenon and the observed track will always agree with the classical result within experimental error, even for arbitrary potential field (containing beyond quadratic terms), in violation of quantum mechanical prediction (Ehrenfest's theorem). This testable prediction offers an interesting experimental verification, crucial for the understanding of the nature of quantum–classical relationship.

Also, analysis of wavepacket spreading calculation shows that for particles with mass $\sim 10^{-12}$ g, the insignificant spreading will always be stymied by infinitesimal environmental disturbances, ruling out any possibility of interference. This result may help in identifying the boundary region separating the quantum domain of low mass and the classical domain of large mass.

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