On the use of biorthogonality relations in the solution of some boundary value problems for the biharmonic equation*

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Let a function $\psi(x,y)$, biharmonic in the semi-infinite strip $\{-1/2 < x < 1/2,\ y < 0\}$, be such that the function and its normal derivative vanish on the side walls $x = \pm 1/2$. We consider the problem of determining this function when we are given $\psi(x,0)$ and $\nabla^2\psi(x,0)$ on the short edge $y = 0$. First, we give a direct method of obtaining a biorthogonality relation among the eigenfunctions and then give a formal solution of the boundary value problem using this relation. Next we show that if we attempt to use this solution using a finite number of terms of the series, it is inferior to a solution where the expansion coefficients are calculated using a least squares procedure. This is a surprising result considering that for Fourier series, the Fourier coefficients are always optimal.

THE biharmonic equation

$$\nabla^4\psi(x,y) = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\psi(x,y) = 0, \quad (1)$$

arises when solving problems in a number of areas of science and engineering, including elasticity and slow fluid flow. Since the equation is linear, it is natural to seek separable solutions appropriate to the given geometry and then superpose these to satisfy the given boundary data. Whereas when seeking harmonic functions, i.e. those that satisfy Laplace's equation, one is led to essentially real Fourier series, here one is led to series of biharmonic functions, the Papkovich–Fadle eigenfunctions, which are essentially complex. In the former case one has the orthogonality property of the trigonometric polynomials, a consequence of the self-adjointness of the reduced Laplace operator, to uniquely determine the expansion coefficients. On the other hand, in the latter case there are only special forms of the boundary data, those corresponding to the so-called canonical problems, for which biorthogonality (BO) relations hold among the eigenfunctions which permit the complex expansion coefficients to be determined analytically.

The purpose of the present article is to display an easy, straightforward route to the BO relation in a canonical problem and to point out a surprising behaviour of the resulting series solutions. Although it is unlikely that there is anything new in this article, it is almost certain that these results are not widely known even among the practitioners.

A biorthogonality relation

The region considered here is the semi-infinite strip $\{-1/2 \le x \le 1/2,\ y \le 0\}$. We will be seeking functions $\psi(x,y)$ satisfying eq. (1) in its interior and such that ψ and $\partial\psi/\partial x$ vanish on $x = \pm 1/2$. If we now assume separable solutions of the form $\phi(x)e^{\lambda y}$, where λ is a complex scalar, $\phi(x)$ will have to satisfy the reduced biharmonic equation

$$\left(\frac{d^4}{dx^4} + 2\lambda^2\frac{d^2}{dx^2} + \lambda^4\right)\phi(x) = \left(\frac{d^2}{dx^2} + \lambda^2\right)\left(\frac{d^2}{dx^2} + \lambda^2\right)\phi(x) = 0. \quad (2)$$

The factorization in eq. (2) suggests that we write $u = \phi'' + \lambda^2\phi$; the equation can then be written as the pair of equations

$$u = \phi'' + \lambda^2\phi,\ u'' + \lambda^2 u = 0, \quad (3)$$

where the primes denote differentiation with respect to x. Rewrite this as follows

$$-\frac{d^2}{dx^2}\phi + u = \lambda^2\phi, \quad (4)$$

$$-\frac{d^2}{dx^2}u = \lambda^2 u. \quad (5)$$

Now, if we let U be a two-vector defined by $U^T = (\phi,u)$ (ref. 1), eqs (4) and (5) are now of the standard form:

$$LU = \lambda^2 U, \quad (6)$$

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where the operator L is given by

$$L=\begin{pmatrix} -\dfrac{d^2}{dx^2} & 1 \\ 0 & -\dfrac{d^2}{dx^2} \end{pmatrix}. \tag{7}$$

If $U^T = (\phi,u)$ and $V^T = (\chi,v)$ are a pair of two vectors, their inner product is now defined in the usual manner

$$\langle V,U\rangle = \int_{-1/2}^{1/2}\{\phi(x)\chi(x)+u(x)\,v(x)\}\,dx. \tag{8}$$

For such pairs whose components also satisfy the boundary conditions $\phi(\pm 1/2) = \phi'(\pm 1/2) = v(\pm 1/2) = v'(\pm 1/2) = 0$, a straightforward computation shows that

$$\langle V, LU\rangle = \langle U, L^*V\rangle, \tag{9}$$

where the adjoint operator L^* is given by

$$L^*=\begin{pmatrix} -\dfrac{d^2}{dx^2} & 0 \\ 1 & -\dfrac{d^2}{dx^2} \end{pmatrix}. \tag{10}$$

Note that to derive eq. (9), the boundary conditions had to be used to eliminate the boundary terms arising from an integration by parts. Now, if U is an eigenvector of L corresponding to the eigenvalue λ as in eq. (6) and V is an eigenvector of L^* corresponding to the eigenvalue μ, i.e. $L^*V = \mu^2 V$, eq. (9) implies

$$\begin{aligned}\langle V,LU\rangle &= \langle V,\lambda^2 U\rangle = \lambda^2\langle V,U\rangle = \langle U,L^*V\rangle \\ &= \langle U,\mu^2 V\rangle = \mu^2\langle U,V\rangle.\end{aligned} \tag{11}$$

It follows from eq. (11) that $(\lambda^2 - \mu^2)\,\langle V,U\rangle = 0$ and so if $\mu \neq \lambda$, $\langle V,U\rangle = 0$. This is the BO relation that was sought. It should be noted how easily and directly it follows from the factorization of the reduced biharmonic operator in eq. (2). This may be compared with the somewhat artificial derivation given, for example, in Joseph[2].

We will use this BO relation in the next section to formally solve a boundary value problem for the biharmonic equation.

Series solutions for a canonical boundary value problem

Suppose that we wish to solve eq. (1) in the semi-infinite strip given that (i) $\psi(x,y)$ and its normal derivative vanish on $x = \pm 1/2$, (ii) $\psi(x,y) \to 0$ as $y \to -\infty$, and (iii) it is subject to the short-edge boundary conditions

$$\psi(x,0) = p(x), \quad \nabla^2\psi(x,0) = q(x), \tag{12}$$

where $p(x)$ and $q(x)$ are sufficiently smooth; we shall also assume the compatibility conditions $p(\pm 1/2) = p'(\pm 1/2) = 0$. This formulation bears, for example, on the problem of the bending of a semi-infinite strip under appropriate edge conditions[3], and on the problem of the creeping flow of a liquid with a free surface[4].

In order to keep matters simple, let us assume here that $p(x)$, $q(x)$ and $\psi(x,y)$ are all symmetric in x; this is no limitation as the antisymmetric case can be handled in a similar way and the general case can be handled by a superposition of the two. In this case the vector eigenfunction $U^T(x;\lambda) = (\phi(x;\lambda), u(x;\lambda))$ is given by

$$\phi(x;\lambda) = x\sin\lambda x + b_\lambda\cos\lambda x, \quad u(x;\lambda) = 2\lambda\cos\lambda x, \tag{13}$$

where $b_\lambda = -1/2\,\tan\lambda/2$ and the eigenvalue λ satisfies the equation

$$\sin\lambda = -\lambda. \tag{14}$$

All the eigenvalues are complex and if λ is an eigenvalue, so are $-\lambda$ and $\bar{\lambda}$. We order all eigenvalues in the first quadrant in increasing values of their real parts; the first two eigenvalues, for example, are approximately $\lambda_1 = 4.21239 + 2.25073i$ and $\lambda_2 = 10.71254 + 3.10315i$.

It is a simple matter to verify that the vector eigenfunction $V^T(x;\mu) = (\chi(x;\mu), v(x;\mu))$ corresponding to the eigenvalue μ of the adjoint L^* is given by

$$\chi(x;\mu) = 2\mu\cos\mu x, \quad v(x;\mu) = x\sin\mu x + b_\mu\cos\mu x, \tag{15}$$

where $v(x;\mu)$ satisfies the reduced biharmonic equation and μ satisfies eq. (14). The BO relation derived earlier guarantees us that if $\mu \neq \lambda$, $\langle V,U\rangle = 0$. On the other hand, a direct calculation shows that if $\mu = \lambda$

$$\langle V(x;\lambda), U(x;\lambda)\rangle = -2\cos^2\frac{\lambda}{2}. \tag{16}$$

We are now in a position to write down a formal solution to the boundary value problem posed above using the BO relation. Let us write

$$\begin{aligned}\begin{pmatrix} p(x) \\ q(x)\end{pmatrix} &= \sum_\lambda A_\lambda U_\lambda = \sum_\lambda A_\lambda \begin{pmatrix}\phi(x;\lambda) \\ u(x;\lambda)\end{pmatrix} \\ &= \sum_\lambda A_\lambda \begin{pmatrix} x\sin\lambda x + b_\lambda\cos\lambda x \\ 2\lambda\cos\lambda x\end{pmatrix},\end{aligned} \tag{17}$$

where the sums are over all eigenvalues with positive real part alone; this is to ensure boundedness for large negative y. Note that the sums in eq. (17) yield real functions, since the eigenvalues appear as conjugate pairs; also A_λ are complex scalars that have to be determined from the boundary data. If we now take the inner product of eq. (17) with $V(x;\mu)$, all the terms on the right vanish, except for the one corresponding to $\lambda = \mu$. Thus we get an explicit expression for the scalars

$$\begin{aligned}A_\lambda = -\frac{2\lambda}{\cos^2\frac{\lambda}{2}}\int_{-1/2}^{1/2}&[p(x)\,\{2\lambda\cos\lambda x\} + q(x) \\ &\{x\sin\lambda x + b_\lambda\cos\lambda x\}]\,dx.\end{aligned} \tag{18}$$

For theorems on the conditions under which biorthogonal series such as eq. (17) converge, see refs 2, 5 and 6; for completeness theorems related to the Papkovich–Fadle eigenfunctions see, for example, ref. 7.

How good is this solution? It is known that if one insists on solutions that are well-behaved at the corners, the solution to the problem is unique; thus if the series in eq. (17) converge to the boundary values, they must represent the unique solution. We consider two cases here: Case 1 with $p(x) = (1/4 - x^2)^2$, $q(x) = 0$ and Case 2 with $p(x) = 0$, $q(x) = 1$. Note that in Case 1 $p(1/2) = p'(1/2) = 0$, and so the compatibility conditions are satisfied in both cases. We will compare the results of calculations based on eqs (17) and (18) with those using a least squares (LS) procedure to determine the coefficients A_λ (ref. 8). It was the expectation of the author, based on the behaviour of Fourier series and on the general high regard that one normally has for explicit formulae, that the BO procedure would be superior.

Figure 1 *a* and *b* displays the results of calculations for Case 1 using ten terms in eq. (17), while Figure 1 *c* and *d* shows the results using 100 terms. In the former case, BO

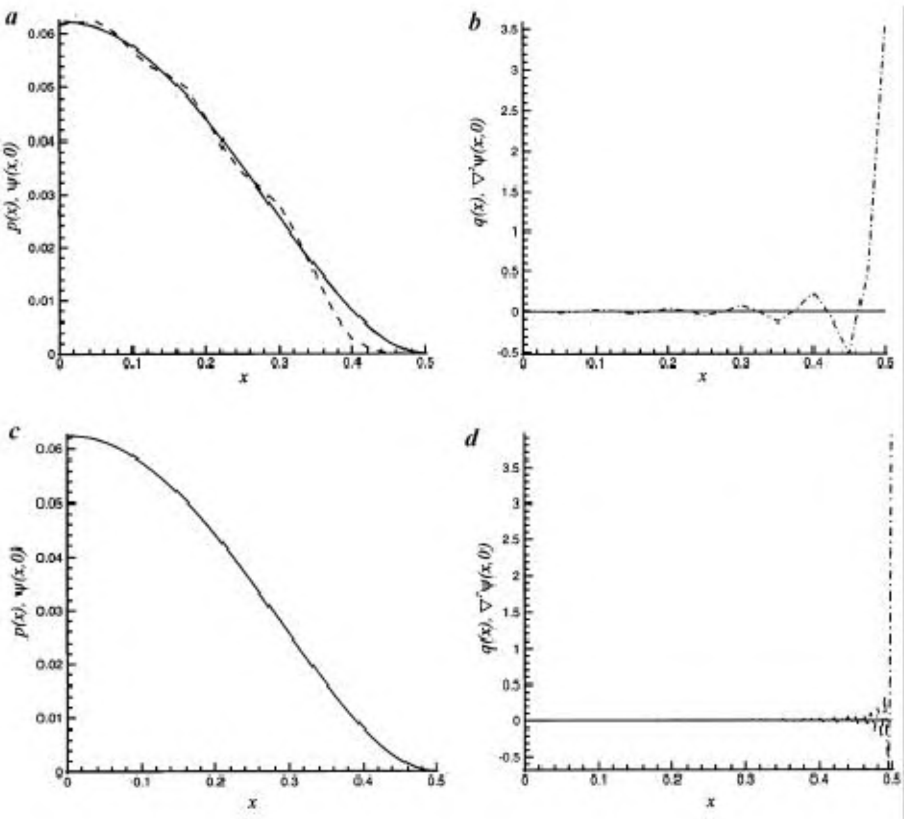

Figure 1. Results of calculations for Case 1 with $p(x) = (0.25 - x^2)^2$ and $q(x) = 0$. ***a*** and ***b***, $N = 10$; ***c*** and ***d***, $N = 100$. ————, $p(x)$ and $q(x)$; ----------, Least squares; - · · - · · -, Biorthogonality.

Table 1. Complex expansion coefficients for Case 1 with $p(x) = (1/4 - x^2)^2$, $q(x) = 0$

	Least squares		Biorthogonality
n	$N = 10$	$N = 100$	A_{λ_n}
1	0.82131E–02 + 0.58585E–01*i*	.10084E–01 + .58431E–01*i*	.10090E–01 + .58428E–01*i*
2	0.13836E–01 – 0.28566E–02*i*	.11388E–01 – .39122E–02*i*	.11380E–01 – .39077E–02*i*
3	– 0.62850E–02 – 0.25690E–02*i*	–.42270E–02 – .53792E–04*i*	–.42162E–02 – .59127E–04*i*
4	0.25757E–02 + 0.40177E–02*i*	.19791E–02 + .36672E–03*i*	.19663E–02 + .37233E–03*i*
5	0.33117E–03 – 0.36354E–02*i*	–.10916E–02 – .31967E–03*i*	–.10770E–02 – .32515E–03*i*
6	– 0.22230E–02 + 0.17665E–02*i*	.67372E–03 + .24653E–03*i*	.65732E–03 + .25154E–03*i*
7	0.20200E–02 + 0.10154E–02*i*	–.45114E–03 – .18819E–03*i*	–.43314E–03 – .19247E–03*i*
8	0.22989E–03 – 0.18793E–02*i*	.32153E–03 + .14575E–03*i*	.30204E–03 + .14904E–03*i*
9	– 0.73306E–03 – 0.45717E–03*i*	–.24081E–03 – .11530E–03*i*	–.21997E–03 – .11739E–03*i*
10	– 0.95548E–04 + 0.62707E–04*i*	.18782E–03 + .93368E–04*i*	.16579E–03 + .94053E–04*i*

does better as regards representing $p(x)$ but fares worse, much worse, as regards representing $q(x)$; in the latter case, both do well as regards $p(x)$ but as regards $q(x)$, BO shows improvement. The errors are still very large towards the end point and it is not clear whether they will improve with more terms. There is no doubt that LS is far superior. A similar situation exists in Case 2, shown in Figure 2, where once again BO fares poorly.

What is going on here? It appears that the convergence of the BO series is not uniform in x. The number of terms required for given accuracy depends on the position on the interval; this is happening when the boundary data are so smooth, whereas in applications we can expect more badly behaved data. Another puzzling and disturbing doubt is that if two methods of evaluating the coefficients A_λ lead to such apparently different results, what has happened to uniqueness? In fact, what had been expected was that the LS coefficients would approach the BO coefficients as the number of terms of the series used, N, became large. As can be seen from Tables 1 and 2 this is, indeed, happening. The position is that if A_{λ_n} is the nth coefficient as given by BO, then the nth coefficient given

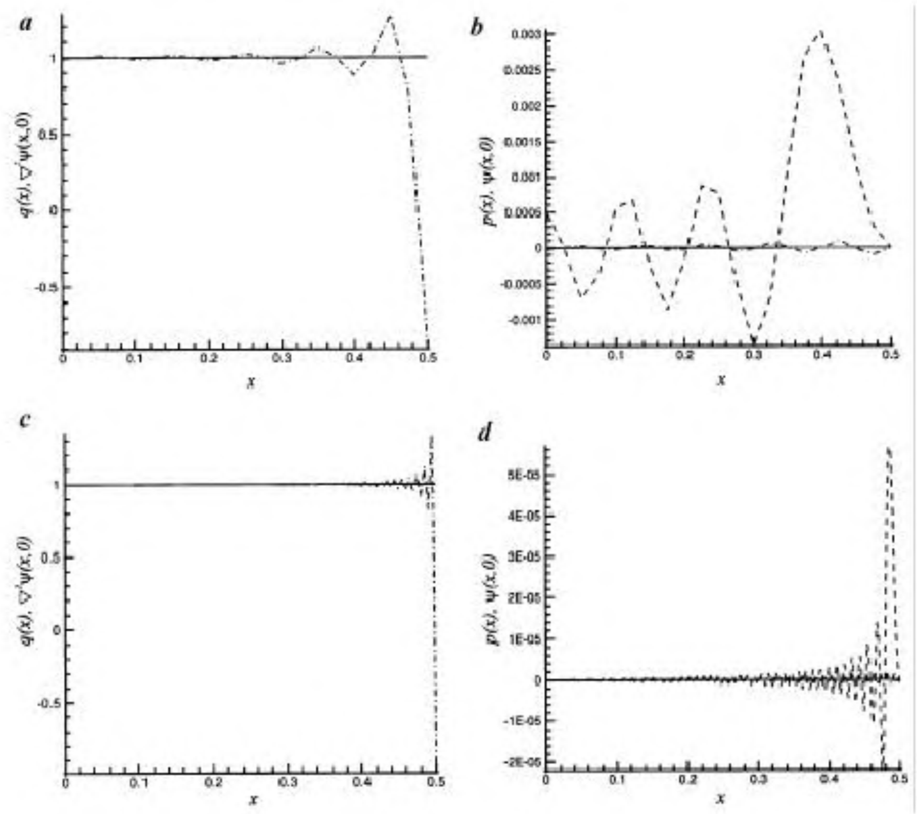

Figure 2. Results of calculations for Case 2 with $p(x) = 0$ and $q(x) = 1$. ***a*** and ***b***, $N = 10$; ***c*** and ***d***, $N = 100$. ———, $p(x)$ and $q(x)$; ---------, Least squares; - · · · - · -, Biorthogonality.

Table 2. Complex expansion coefficients for Case 2 with $p(x) = 0$, $q(x) = 1$

n	$N = 10$	$N = 100$	A_{λ_n}
1	.66146E–01 – .10866E–01i	.65060E–01 – .10781E–01i	.65056E–01 – .10779E–01i
2	–.83334E–02 – .21050E–02i	–.69188E–02 – .14701E–02i	–.69144E–02 – .14724E–02i
3	.32992E–02 + .22777E–02i	.21292E–02 + .78934E–03i	.21237E–02 + .79206E–03i
4	–.12445E–02 – .25825E–02i	–.95099E–03 – .44085E–03i	–.94445E–03 – .44371E–03i
5	–.38300E–03 + .21780E–02i	.51697E–03 + .27068E–03i	.50949E–03 + .27347E–03i
6	.14279E–02 – .98960E–03i	–.31796E–03 – .17903E–03i	–.30959E–03 – .18158E–03i
7	–.12260E–02 – .69015E–03i	.21313E–03 + .12536E–03i	.20393E–03 + .12753E–03i
8	–.17415E–03 + .11624E–02i	–.15236E–03 – .91845E–04i	–.14241E–03 – .93514E–04i
9	.45916E–03 + .29602E–03i	.11457E–03 + .69885E–04i	.10394E–03 + .70938E–04i
10	.62006E–04 – .39341E–04i	–.89767E–04 – .54968E–04i	–.78521E–04 – .55304E–04i

by LS does indeed tend to A_{λ_n} as $N \to \infty$. However, for a fixed N the LS procedure approximates the boundary conditions much better than if the asymptotically correct coefficients were used. This means that for finite N, it is best to use LS. This is an unexpected result.

Conclusion

We have suggested a direct method for obtaining the BO relation for a canonical boundary value problem for the biharmonic equation. This simple method depends crucially on the possibility of factorizing the reduced operator. We believe that the method will be applicable in other situations where the underlying operator can be factorized.

Calculations with two sets of boundary data have shown that if a finite number of terms of the series is used, the LS procedure yields solutions that are much more accurate than those that would be obtained using the coefficients given by the BO procedure. This is quite different from what obtains in the case of Fourier series. Let ϕ_n be orthonormal on $[a, b]$; let c_n be the nth Fourier coefficient of $f(x)$ relative to ϕ_n, i.e.

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}\,\mathrm{d}t$$

and let

$$s_n(x) = \Sigma_1^n c_m \phi_m(x)$$

be the nth partial sum of the Fourier series of f. Let $t_n(x) = \Sigma_1^n \gamma_m \phi_m(x)$. Then by Theorem 8.11 of Rudin[9]

$$\int_a^b |f - s_n|^2\,\mathrm{d}x \le \int_a^b |f - t_n|^2\,\mathrm{d}x,$$

and equality holds if and only if $\gamma_m = c_m$, $m = 1, 2, \ldots$. In other words, the Fourier coefficients lead to the best possible partial sum representation of f in the mean square sense. A similar result does not seem to hold when biharmonic eigenfunctions are used to represent biharmonic functions in the rectangle.

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Received 28 August 2003