The Navier–Stokes Equations – An Elementary Functional Analytic Approach. Hermann Sohr. Birkhäuser Verlag, P. O. Box 133, CH-4010 Basel, Switzerland. 2001. 367 pp. Price not mentioned.

Even after almost two centuries of intense effort on the part of many mathematicians, engineers and scientists, the Navier–Stokes equations

$$u_t - \nu\Delta u + u.\nabla u + \nabla p = f, \quad (1)$$
$$\text{div } u = 0,$$

have continued to charm, intrigue and baffle their many devotees. Whether or not one agrees with the oft-quoted statement that turbulence is the last unsolved problem of classical physics, one will certainly admit that we are still far from solving these equations in any generality, even in fairly simple situations; and we continue to be surprised by new features that crop up in more complex situations.

The Navier–Stokes equations are those that govern the motion of an incompressible liquid having internal friction or viscosity. These equations, discovered independently by Navier (1822) and Stokes (1845), are a mathematical formulation of Newton's Second Law of Motion (rate of change of momentum $\propto$ external force) extended to a liquid with internal friction; strictly, they hold only when the internal stress is linearly proportional to the strain rate. In eq. (1), u is the liquid velocity vector field, f the external force field and p the scalar pressure field, all in a domain Ω in a Euclidean space R^n, ν is the kinematic viscosity, an internal fluid friction parameter. Typically, given f, we might want to deduce u and p, with say u vanishing on the boundary $\partial\Omega$. Now, from a practical engineer's point of view, the equations are important because where ever liquid motions occur, they come into play. Thus, they bear on the flow of oil in pipelines, the movement of air in the lungs, in estimating the drag on a car or train or the power consumed by a pump. For the mathematician or theoretician, the equations are interesting and challenging because they are intrinsically nonlinear and are a source of beautiful examples of singular perturbations, bifurcations and other manifestations of nonlinearity.

It is an unfortunate, but perhaps inescapable fact that the different groups that use and work on the Navier–Stokes equations have little to do with one another. Practical engineers use empirical rules based on past experience, approximate solutions and numerical computations to 'solve' the practical fluid problems that arise in practice. Engineering scientists, applied mathematicians and physicists try to get exact and rational approximate solutions to very specific problems usually involving idealized geometries. It has been pointed out that fluid mechanics is a field in which knowledge of the field is obtained only from the slow accumulation of specific examples. Finally, mathematicians are interested in the existence, uniqueness and smoothness of solutions to eq. (1) in arbitrary domains and dimensions. Although the three groups could obviously benefit by interacting with one another, this rarely happens because their interests and vocabularies appear so different. The mathematical works appear so difficult and impenetrable and the final results so 'useless', that most non-mathematicians completely ignore the mathematical literature on the Navier–Stokes equations. Under these circumstances, any attempt to bridge this divide must be applauded.

The book under review by Hermann Sohr, who has himself contributed to the mathematical literature, has as its objective the development of 'an elementary and self-contained approach' to the mathematical theory of the Navier–Stokes equations in a domain in Euclidean space. One should not be fooled by the phrase in quotation marks: in order to even browse through this book one would need a reasonable familiarity with real analysis and the basic tools in Hilbert and Banach spaces. The book consists of five chapters entitled: (i) Introduction, (ii) Preliminary results, (iii) The stationary Navier–Stokes equations, (iv) The linearized nonstationary theory and (v) The full nonlinear Navier–Stokes equations. In the first chapter, after introducing the Navier–Stokes equations and some notation, an overview of the functional analytic approach is presented, which is in fact a summary of the last three chapters. In particular, the Stokes operator is introduced, which is used in all the later chapters. Surprisingly, but typical of the book, the definition of the various spaces that are needed, the L^q-spaces, the spaces of distributions and the Sobolev spaces, are given *after* the overview, where they are already used; if the order had been reversed, the novice reader would suffer less of a shock. A similar idiosyncrasy occurs in chapter II. The useful section on the basic facts of Banach spaces, Hilbert spaces, the Laplace operator and the resolvent *follows* the earlier sections on embedding properties and the nabla and div operators. This chapter has a large number of lemmas, the proofs of many of which are left to the references. The fact of the matter is that a great deal of preliminary material has to be mastered before we can attempt to tackle the Navier–Stokes equations.

The steady Navier–Stokes equations, where the time is absent, are studied in chapter III. Sohr begins with a discussion of 'weak solutions' of the Stokes equations

$$-\nu\Delta u + \nabla p = f, \text{div } u = 0, \quad (2)$$
$$u|_{\partial\Omega} = 0.$$

Note that these are linear equations obtained from the steady version of eq. (1), by dropping the nonlinear convective terms $u.\nabla u$. In introducing the concept of weak solutions, Sohr says 'The idea is the following: It seems to be rather difficult to prove directly the existence of classical regular solutions. Therefore, we argue indirectly. In the first step, we get rid of the pressure p and construct a so-called weak solution using a Hilbert space argument. In the second step, we construct the pressure p and prove regularity properties of u, p under smoothness assumptions on f and Ω'. This is the strategy used throughout the book: determine a 'weak solution' (in an appropriate Sobolev space) which satisfies an auxiliary equation without the pressure, construct an associated pressure so that the u and p now satisfy eq. (2) or the relevant equation in a distributional sense, and finally prove regularity under appropriate assumptions. Whereas the analysis of the Stokes equations is carried out for $n \geq 2$, for the stationary Navier–Stokes equations the nonlinearity forces the author to restrict himself to the cases $n = 2, 3$.

Although the overall strategy remains the same in the linearized nonstationary theory of chapter IV, naturally, some new spaces and operators need to be introduced to handle the integrations in time. Finally, the full Navier–Stokes equations (eq. (1)) are dealt with in chapter V. Here, once again, n is restricted to be 2 or 3. As before, a weak solution has to be defined, along with the auxiliary pressureless equation that it satisfies, and the same general procedure has to be followed, but with additional hurdles posed by the nonlinearity. Both in the linearized nonstationary theory and the case of the full Navier–Stokes equations, certain energy equalities and inequalities appear to play a significant role. In the latter case, there are differences even between the two-dimensional and three-dimensional cases: sometimes additional conditions are required in the $n = 3$ case to prove the existence of weak solutions. We note that theory is developed for a completely general domain Ω. This means that it applies to arbitrary, unbounded, non-smooth domains. It is on this account that the restriction to the lower dimensions appears in the nonlinear case.

I should point out some of the strengths and weaknesses of the book. Although a fair amount of background is needed on the part of the reader, there is no doubt that the author has attempted to present a unified and self-contained account of the theory. The book is well written and not unnecessarily wordy. There is an up-to-date bibliography and a nice index. And where proofs are given, there are enough details that a reader with the proper background will be able to follow the argument. In my opinion, the author has succeeded in what he set out to do. My main complaint is that often the proofs and the lines of argument do not seem to be properly motivated. For example, when a weak solution is defined, no reasons are given as to why that definition has been chosen. When a particular space is chosen for some variable, we are not told the basis for this choice. This will certainly be a stumbling block for the non-expert. Another small problem with the book is that at times, as pointed out earlier, the order of presentation is not the natural order.

Who would benefit from reading this book? Certainly, a mathematician who wishes to know what the important issues concerning eq. (1) are and what has been achieved, would find this an excellent source. Equally, a mathematically-minded student, with a good grounding in analysis and who has decided to work in this area, or the teacher who wants to teach a course on this material would find this a valuable text. Not so obviously, the book would be of use to a dedicated teacher of analysis or functional analysis, who wishes to show his students that analysis really does have applications outside pure mathematics. The closed graph theorem, the Fisher–Riesz theorem, the fixed point principle, Fubini's theorem, the Hahn–Banach theorem, Hölder's inequality, the Leray–Schauder principle, the Riesz representation theorem and many other classic theorems are routinely used in this work. Will not a bright young student be better motivated to study analysis, if he sees the 'practical' use of what he is learning? I will now conclude with a possibly even more shaky suggestion. I think there are ideas in this book which may possibly suggest certain methods of actually obtaining rational or good approximate solutions to specific fluid-dynamic problems. To derive this benefit, however, one would have to take the trouble to learn at least the rudiments of a different language. It may well be worth it. I think Sohr has built a bridge that connects the practising fluid dynamicists to the mathematical fluid dynamicists and, hopefully, it will be used.

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Ecological Stoichiometry: The Biology of Elements from Molecule to the Biosphere. Robert W. Sterner and James J. Elser. Princeton University Press, 41 William Street, Princeton, NJ 08540, USA. 2002. 439 pp. Price: US$ 29.95.

Ecology as a science was once thought not refutable by the criterion of conjecture and refutation of Karl Popper; some even thought it was a weak science full of tautologies and circular reasoning. Any scientific theory should provide for a method to test it. In physics, the same notion prevailed about relativity – a remarkable theory; but when proposed, not enough experimental proof was available. However, one is not free to propose any type of theory – by any stretch of imagination and hope it would be proved right some day. In physics, experimental proof may be delayed, but a theory may be popular because of strong mathematical proof that may exist for it. These arguments are there in the book under review but not in sufficient detail. For example, the classic case often cited but not found in the book, is Clements concept of a climatic climax. Clements predicted that ecological succession always leads to a mono-climax determined by a particular kind of climate.

Ecological succession always ended in a form of dominant vegetation representing the mono-climax. If a mono-climax were not found, Clements would have said: 'If we wait long enough we would get it'. This theory of mono-climax is a weak one because it gets expanded to accommodate observations not predicted by it when first proposed. May be the next edition of the book can take care of such serious lacunae in the first chapter. Despite all this, one cannot help falling in love with the book for its sheer clarity and directness, and the refined experimental approach to the problem of ecological stoichiometry. The ideas have been presented lucidly from the point of concepts and definitions.

There have been attempts to develop the ideas and the main theme of the book from as far back as 1913, when Henderson published his *Fitness of the Environment*. It was considered a classic then on the elemental composition of living things. The first work on ecological stoichiometry as a concept proper was by Redfield in 1958, which became eponymous as Redfield ratios, while the latest on the subject is Reiner's in 1986. The book has its relevance from many other angles too, be it global warming or the greenhouse effect, the stable concentrations of CO_2 and O_2 in our atmosphere, nutrient cycling in aquatic ecosystems and many more. Reading the book has been like going through a journey where landscapes generate a veritable kaleidoscope of ideas and concepts. Each organism could be viewed as a stable steady state, either converting O_2 to CO_2 or vice versa. After all, for the ecosystem homeostasis is just the resilience of a system. Each ecosystem, including the whole biosphere, could be viewed similarly. The concepts of energetics of ecosystem and non-equilibrium thermodynamics developed by Ilya Prigogine could be applied to them, as amply exemplified in chapter 7 of the book under review. The book abounds in conceptual models that can be employed in physiology and evo-