Supersymmetry in Quantum Mechanics. Fred Cooper, Avinash Khare and Uday Sukhatme. World Scientific, Singapore. 2001. Price: US $ 58. 210 pp.

In 1964 Gürsey and Radicati, and independently Sakita, proposed spin–unitary-spin symmetry $SU(6)$ for hadrons. This was patterned on the nonrelativistic spin-isospin symmetry $SU(4)$ proposed by Wigner for nuclear physics. The $SU(6)$ symmetry was eminently successful in explaining the observed spectrum of low-lying mesons and baryons, their masses (Bég-Singh mass formulae) and their magnetic moments (Bég-Lee and Pais). In view of this impressive success there was feverish activity to find a relativistic generalization of $SU(6)$. Eventually it was realized by McGlinn, O'Raifeartaigh and others that it is not possible to combine internal symmetries with Poincaré symmetry (relativistic invariance) in a non-trivial way. The best result of this kind is that given by Coleman–Mandula theorem. In all these papers it was assumed that the internal symmetry generators were bosonic. Nobody in those days could even imagine that they could be anything else. If the possibility that internal symmetry generator can be fermionic had been entertained at that time then one would have discovered supersymmetry in late nineteen sixties. In view of this missed opportunity, the supersymmetry was first discovered only in 1971 in the context of string theory in the guise of 'World-sheet supersymmetry', by Gervais and Sakita. It was later extended to four-dimensional field theories by Wess and Zumino in 1974. Independently it was also discovered in Russia by Gol'fand and Likhtman in 1971 and by Volkov and Akulov in 1973, but these papers did not catch the attention of the physics community. Supersymmetry is now strongly believed to be an important ingredient in our understanding of high energy physics. It is believed that a unified description of all the fundamental interactions would require the existence of supersymmetry.

Exact supersymmetry would require for each known fermion a bosonic partner with the same mass and vice versa. Since this is not observed, the supersymmetry must be dynamically broken. In supersymmetric theories, the Hamiltonian is the sum of squares of various supersymmetric charges. Thus energy is automatically a semipositive definite for all the states. If a vacuum state with lowest possible energy i.e. zero exists, then it must be annihilated by all the supercharges acting on it, i.e. the vacuum state is fully supersymmetric. Since the symmetry of the vacuum is the symmetry of the theory, it follows that the supersymmetry is unbroken. This looked like an impasse. In order to understand how a dynamical breakdown of supersymmetry could possibly occur, Edward Witten introduced in 1981 a simplified model, 'supersymmetric quantum mechanics'. The model has two supercharges Q and $Q^\dagger$, which obey $Q^2 = Q^{\dagger 2} = 0$, and the Hamiltonian $H = QQ^\dagger + Q^\dagger Q$. More explicitly, with $Q = Q_1 + iQ_2$ we can realize the system through the following choice $\left(p = \frac{1}{i}\partial/\partial x\right)$

$$Q_1 = \frac{1}{2}(\sigma_1 p + \sigma_2 W(x)),$$

$$Q_2 = \frac{1}{2}(\sigma_2 p - \sigma_1 W(x)),$$

$$\begin{aligned} H &= Q_1^2 + Q_2^2 \\ &= \tfrac{1}{2}\left(p^2 + W^2 + \sigma_3 \tfrac{\mathrm{d}W}{\mathrm{d}x}\right), \end{aligned}$$

where σ_1, σ_2, σ_3 are the usual Pauli matrices. A study of this model showed that when the supersymmetry is dynamically broken, the ground state energy is nonzero and positive. This model is related to potential theory in nonrelativistic quantum mechanics with potentials $V_\pm(x)$ given by

$$V_\pm(x) = \tfrac{1}{2}\left(W^2 \pm \tfrac{\mathrm{d}W}{\mathrm{d}x}\right).$$

The biggest impact of supersymmetric quantum mechanics has been in the area of exactly solvable potential models. A crucial ingredient in this has been the concept of 'shape invariant' potential introduced by Gendenshtein in 1983. If the potentials $V_\pm(x)$ are related to each other by

$$V_+(x, a_1) = V_-(x, a_2) + R(a_1),$$

where a_1, a_2 are sets of parameters, then these potentials are called 'shape invariant'. The problem of general characterization of all such potentials with arbitrary relationship between the set of parameters a_2 and a_1 is still unsolved. If the parameters a_2 are related to the parameters a_1 by a translation, then we already obtain all the usual well-known solvable potentials such as shifted oscillator, three-dimensional oscillator, Coulomb, Morse, Scarf I (trigonometric), Scarf II (hyperbolic), Rosen–Morse I (trigonometric), Rosen–Morse II (hyperbolic), Eckart and Pöschl–Teller potential; quite an impressive list. Another solvable case for the potentials is when the parameters a_2 and a_1 are related by scaling. More general relationships have been only investigated partially so far. The method of shape invariant supersymmetric potentials also throw light on earlier methods like the factorization method of Infeld and Hull.

In the first three chapters of this monograph the authors describe preliminaries relating to one-dimensional Schrödinger equation and factorization of a general Hamiltonian after a brief introduction. The shape invariance is treated in the fourth chapter. These techniques are then applied to the problem of charged particles in external electromagnetic fields. The method of supersymmetry proves quite powerful to obtain isospectral Hamiltonians, i.e. to construct a n-parameter family of potentials having bound states, and scattering data, identical to a given potential. This is related to exact multi-soliton solution of a number of nonlinear equations such as Korteweg-de Vries equation. Apart from Schrödinger Hamiltonians having discrete and/or continuum spectra, the authors discuss the case of periodic potentials having band spectra. They discover a whole new set of periodic potentials which have exactly solvable band spectrum. The remaining two chapters treat approximation methods in supersymmetric quantum mechanics, viz. supersymmetric WKB approximation and perturbation theory.

The authors of this monograph have made significant contributions to this field, especially those related to systematics of shape invariant potentials, new exact models of band structure and WKB approximation. They have produced a delightful monograph on the subject. The usefulness of this monograph is further enhanced by a set of exercises given with each chapter.

VIRENDRA SINGH

Department of Theoretical Physics,
Tata Institute of Fundamental Research,
Mumbai 400 005, India
e-mail: vsingh@theory.tifr.res.in