Mathematics – Its place and role in our society*

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'Mathematics is the queen of Sciences.' Those are the words of Carl Frederick Gauss. Much the same sentiment is expressed in the ancient Sanskrit verse:

यथा शिखा मयूराणां नागानां मणयो यथा ।
तथा वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम् ॥

Gauss was, of course, one of the greatest mathematicians of all time; and the author of the couplet too was in all probability a mathematician; and their perceptions in this matter may well be at variance with those of outsiders. It is my hope that what I have to say here will help you a little to understand what underlies the mathematicians' claim for their field.

'What is mathematics?' The answer to this question is of course complex; there are elaborate elucidations, some excellent, on the subject but inevitably, even the best accounts give incomplete answers. Any attempt to answer that question will necessarily involve a substantial discussion on the role and place of mathematics in society.

Abstraction is one quality that permeates all mathematics. The first intimations of mathematical activity are no doubt to be found in counting. The act of counting is almost involuntary, but what underlies it is profoundly abstract: the human mind recognizes that there is an attribute that it can ascribe to a collection or set of entities, an attribute which is entirely indifferent to the nature of the individual members of the collection. This attribute is what we call 'the number of entities in the collection'. The collection could be of material objects like fruits in a basket or a more heterogeneous bunch like all the wedding presents received by someone; or insubstantial things like colours in a rainbow or a collection of numbers themselves. Well, I just mentioned a collection of numbers but have not said what a number is. Mathematicians have arrived at some ways of defining the concept, but only recently – in the 20th century. I will briefly describe one definition; and for that one has first to say when two collections have the same number of members. Two collections, say A and B, have the same number of members if one can pair off each member of A with a member of B in such a way that each member of A has exactly one partner in B and vice versa. One definition of a number then is that it is a totality of collections such that any two collections in this totality have the same number of elements. That definition is rather cumbersome and in fact presents certain logical difficulties but other definitions devised cannot escape these problems either.

Abstraction may be at the root of it, but counting owes its discovery to down-to-earth, mundane compulsions: exchange of goods and barter where one had to set values on different commodities. The market place was undoubtedly the principal driving force for all the arithmetic we learn at school. Totalling stock led to addition and multiplication, balancing accounts to the invention of subtraction, sharing assets to division and so on – an affirmation of necessity being the mother of invention. Goaded again by practical needs, geometry, the study of spatial relationships had a parallel development. Indications are that religious rituals contributed a great deal to this. As we know, in the second half of the millennium before Christ, the Greeks invented an entirely new paradigm for presenting geometry – in fact of knowledge in general. I will presently come back to the effects of this revolution, but I want to emphasize now that abstraction was an important component of Euclid's geometry.

Abstraction consists on the one hand in the capacity to zero in on what is relevant in a particular study with an eye to see in what other contexts the same things are relevant and on the other hand, an ability to reject the irrelevant. It took the genius of Galileo to realize that friction is to be treated as irrelevant in the first instance in the study of motion; and that was the first step needed to turn mechanics into a mathematical field. More recent developments reinforce the fundamental role of abstraction in every mathematical endeavour.

I mentioned the revolution effected by the Greeks. It altered not only mathematics but the theory of knowledge itself in a profound way. Euclid's postulational method has definitely influenced our ideas on what parts of knowledge should be labelled as mathematical: one certainly

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considers a body of knowledge as mathematical if it can be fitted into the postulational paradigm of Euclid. That means that the body of knowledge is derived from a certain number of postulates (called axioms by Euclid) which are accepted without argument, through rigorous reasoning, the methods of deduction themselves being governed by set rules which are also to be regarded as postulates.

An amazingly large part of what we regard as mathematical knowledge does meet even this very rigid criterion with only the axioms needed for the elementary arithmetic of natural numbers and just a little more as the foundation. However this last statement has to be tempered by one fact: many mathematical areas which today can be fitted into this remarkably economical postulational scheme, could not meet these stringent demands when they were first apprehended. It is only after Dedekind in the 19th century showed how to extend the number system beyond the rational numbers of elementary arithmetic with only the axioms needed for elementary arithmetic as the starting point, that calculus could be received into the fold of this scheme. Contemporaries of Leibnitz and Newton (inventors of calculus) however, had no hesitation in hailing the birth of some new and wonderful mathematics; and not merely the calculus, a lot of other much less sophisticated mathematics could be derived from the axioms of elementary arithmetic by Euclid's deductive method but only after Dedekind had had his say. Of course Newton and his contemporaries as well as other mathematicians did use the deductive principles *à la* Euclid but based themselves implicitly on a set of much more elaborate unquestioned assumptions than the simple axioms of arithmetic.

Let me give an example of some mathematics of a rather elementary nature that bothered Pythagoras greatly as he was unable to fit it into the framework of arithmetic. His theorem (which incidentally was known to ancient Chinese as well as Hindu mathematicians) says that the hypotenuse of a right-angled triangle whose two other sides are each of unit length has itself a length b whose square is 2. We accept without thought that this b is a 'number', but Pythagoras' subtle mind encountered a difficulty here. He knew the rational numbers of arithmetic, namely numbers of the form p/q, where p and q are integers with q non-zero and those were all the numbers he would admit as numbers. Is b one such number? He discovered that it was not – he argued as follows to come to this conclusion.

Suppose indeed that $b = p/q$ with p and $q \neq 0$ integers. We can reduce this fraction to its simplest form and assume that p and q have no common factors; then one has $2 = b^2 = p^2/q^2$ leading to $2q^2 = p^2$. This means that p^2 is even. Now the square of an odd number is necessarily odd. Thus we conclude that p must be an even number, so is of the form $2r$ for some integer r. This leads to $p^2 = 4r^2 = 2q^2$ and hence $q^2 = 2r^2$. This means that q^2 and hence q is even. But then both p and q are even and so have 2 as a common factor, a contradiction. Thus there cannot be any rational number whose square is 2.

As I said, that worried Pythagoras. But it did not impede the progress of mathematics: one simply assumed that there was an enlarged number system which included the square root of 2 and where one could perform the kind of operations we are familiar with in the arithmetic of rational numbers. Geometry through the idea of the straight line provided an anchor of intuition that offered some security for manipulations with the extended system of numbers – numbers were thought of as points on a line.

The postulational method came in to deep scrutiny in the 20th century. David Hilbert, a towering figure in mathematics, raised some fundamental issues. He proposed that once the axioms and rules are formulated, the entire process of deducing new 'true' statements from the axioms is simply a game and the meanings of such statements have no relevance to the game. If meaning can be given to some of these things, it is to be considered fortuitous but the pursuit of the game is what constitutes mathematical activity. It was however his great hope to be able to formulate an axiom system and rules of deduction that would include the natural numbers and their standard manipulations in its ambit and which could be shown to be internally consistent. But this was shattered by the famous work of the logician Gödel who showed that such an axiom system does not exist. This piece of work has necessarily undermined the mathematician's confidence about the unambiguous nature of the 'truths' he wants to unravel, but despite this, mathematical activity has, since Gödel, grown by leaps and bounds.

Commerce, we saw was the driving force behind the invention of arithmetic, but there are many other human activities that have helped mathematics progress. Physics has had over the years a symbiotic relationship with mathematics. The language of mathematics has repeatedly proved itself to be the most suitable vehicle for understanding physics. Mathematical constructs are found in which one locates concepts and notions that can be put in correspondence with physical entities in a phenomenon one wishes to study. The deduction processes of mathematics can then be used to unveil mutual relationships between the mathematical entities and suggest corresponding connections among the physical entities – what we have on our hands then is a prediction; and if such a prediction is verified to be true, it is indeed a confirmation that the study of the mathematical system is likely to help one understand physical reality; if not, one concludes that no such help is forthcoming and one abandons the mathematical investigation. What has just been described is of course what is called 'Mathematical Modelling' and has been with us since the days of counting; it is somewhat amusing to hear of this being talked about as a major branch of mathematics! The model, that is the mathematical structure is often not quite a finished product (at least not in the postulational scheme) and it is built-up often by drawing heavily on the available knowledge of the physical phenomenon. The classic instance of this is the modelling of mechanics: Calculus was born of this effort. On the other hand, relativity, had in Riemannian geometry, a well-developed mathematical system with which a model could be designed for it. This kind of symbiotic relationship with physics continues to this day.

Many engineering disciplines interact with mathematics in a similar way. The level of sophistication of the mathematics used in dealing with engineering problems has grown by leaps and bounds and with it the ability to handle more and more complex problems. A great deal of the mathematics used in engineering is in the area of differential equations, an off-shoot of the calculus. Probability theory is another area with profound applications to engineering problems. Combinatorics is yet another area that has had an impact on engineering. Till recently, algebraic geometry, a relatively 'pure' domain has been used with considerable success in handling engineering problems.

Biology and medicine which seemed to have practically no use for advanced mathematics at the beginning of the last century, are now benefitting immensely from the intervention of sophisticated mathematical tools. Deep mathematics has made its advent into social sciences as well in recent decades. The work of many of the Nobel laureates in economics is dependent on mathematics. Wall Street employs many mathematics Ph Ds to handle problems of finance.

Computer science is essentially an off-shoot of mathematics, with logic and combinatorics playing a basic role. The revolution in information technology has its roots in mathematics. In sum, mathematics has been a major contributor to human progress in diverse directions. It has also contributed in no small measure to the not-so-lofty business of military development and war itself. It is said that the First World War was the war of the chemist, the Second that of physicist and if there is going to be (God forbid) a third one, it will be the mathematician's!

I have dwelt at some length with the impetus received by mathematics from diverse human endeavours and its role in advancing the cause of such endeavours. Much of the mathematics stemming from interaction with other fields of human activity is labelled applied mathematics (perhaps 'applicable' would be the more appropriate adjective), but there is a large body of mathematics whose creation was driven by purely aesthetic considerations. Mathematicians perceive beauty in many mathematical constructions and experience a great deal of excitement in exploring them further. They pose themselves questions that arise purely from a contemplation of mathematical concepts guided by their aesthetic sensibilities. Beauty in art or music is of course a familiar idea; from the time of Plato, it has been analysed in vague terms such as proportion, balance, harmony, etc. These terms are of no avail if one wants to convey what beauty in mathematics means. The best means of getting it across to the non-mathematician is perhaps to state and prove a theorem due to Euclid which most (if not all) mathematicians consider a 'lovely' piece of mathematics: if you see beauty in it, your aesthetic sensibilities will find the right resonance in the mathematical mind.

Let me now state and prove the Euclid theorem referred to. To state the theorem, it is necessary to recall some elementary concepts. A natural number is called a prime if it is different from 1 and 0 and the only numbers that divide it exactly are itself and 1. The first few primes are 2,3,5,7,11,13,17 . . . . The question that Euclid posed to himself was 'Is the collection of all prime numbers a finite collection?'; and he answered it in the negative. Euclid's reasoning runs as follows.

Suppose that the collection of all the primes is indeed finite. Let $p_1, p_2, \ldots, p_n$ be an enumeration of all the prime numbers. Consider the number N which is obtained by adding 1 to the product of all the prime numbers: $N = p_1, p_2, \ldots p_n + 1$. Since it is larger than all the prime numbers (in the exhaustive finite collection $\{p_1, p_2, \ldots, p_n\}$), N is not a prime. Now if we divide N by any prime number, we get a remainder of 1, so that no prime number can divide N. Since N itself is not a prime, it is exactly divisible by a number different from 1 and itself. Now there are only finitely many numbers different from 1 and N that divide N exactly (any number dividing N exactly is smaller than N). Let m be the smallest of the numbers that divide N exactly but are different from 1 and N. If a number l other than m and 1 divides m exactly, such a number l will be less than m and yet divide N exactly – since m divides N exactly. So we find that m is necessarily a prime, which is absurd since we have seen that no prime can divide N.

That was an example from the Number Theory or 'Higher Arithmetic'. The question was raised merely out of curiosity about numbers – not from any practical considerations. Gauss when he called mathematics the queen of sciences also went on to add that Number Theory was the queen of mathematics. And this branch of mathematics draws practically all its inspiration from internal aesthetic considerations (it is however true that some of its outer reaches have nevertheless found applications). Fourier, an outstanding mathematician of the 19th century (and a friend of Napoleon), admonished his greater contemporary Jacobi for 'trifling with pure mathematics' to which Jacobi responded saying 'a scientist of Fourier's calibre should know that the true end of mathematics is the greater glory of the human mind'. It would appear that most great mathematical minds set greater store by the mathematics that resulted from the aesthetic drive than that which came out of the impetus at the interface with the external world.

In trying to say what mathematics is, we have inevitably touched upon its role in the advancement of other diverse human activities. There is also a somewhat indirect impact that mathematics has had on human affairs. Training in mathematics enables a person to develop his capabilities in logical analysis of situations and helps one think objectively on issues in general. Four centuries before Christ, Plato had recognized the value of mathematics in this direction and suggested that men providing leadership must be well-versed in mathematics. There can be little doubt that mathematics has had an important and pervasive role in human progress.

Is the society in general aware of the importance of mathematics? How does it treat the practitioner of mathematics? Western society seems to have had a good appreciation of the importance of science at large, and mathematics in particular, at least in the modern period. Till the advent of the 19th century, there were very few people who made mathematics their exclusive pursuit, but since that time, mathematicians, even those pursuing mathematics for its own sake have been reasonably well supported in the West. The creation of many universities and other centres of learning and the support extended by the nobility to individual mathematicians bear ample witness to this. There have of course been instances (Abel and Galois, two of the all-time greats in mathematics are the most striking cases) when outstanding men did not get the support they needed, but by and large the tremendous progress made in mathematics in Europe and America is in itself evidence of Western societies' support for science. The erstwhile Soviet Union deliberately set out to promote science in general and mathematics in particular; and this policy had a resounding impact. Moscow produced a great mathematics school with an amazingly large number of gifted mathematicians whose creative achievements as well as scholarship were stupendous. Unfortunately this wonderful school has virtually disintegrated since, thanks to the political upheavals there. The Americans were paying relatively little attention to mathematics in the first half of the twentieth century, but the Soviet space programme's first Sputnik jolted them from their benign indifference into eager support for mathematics. Through the sixties and seventies and even into the eighties, support for mathematics was available on a very generous scale in the US and this had indeed a tremendous effect. It produced an array of brilliant mathematicians and much of the most exciting mathematical developments. It would appear that in recent times however the US is reverting to its old attitudes towards mathematics.

What then of our country? Intellectual activity had certainly taken a back seat for centuries in our country and its first stirrings after the long period of dormancy are to be seen in the Bengal renaissance of the 19th century. In the beginning of the reawakening it was the pursuit of humanities that dominated the scene, but in the early twentieth century, the twin figures of Raman and Ramanujan blazed new trails in science. Raman was an outstanding communicator and his leadership provided immense impetus for the development of physics. Mathematics did not have this advantage: Ramanujan's brilliant career was tragically cut short in its prime. Nevertheless his example inspired a good many people to pursue mathematics. A career in mathematics was of course unattractive in comparison with many others when viewed in terms of the creature-comforts that one could command, but in the first half of the twentieth century there was compensation in the kind of respect that the man of learning was accorded. It must be said that both Raman and Ramanujan received reasonable support from the colonial institutions of that period; in the case of Ramanujan, once the people with the requisite powers were convinced of his extraordinary talent, they acted with an alacrity that today's bureaucrats would do well to emulate. Of course Britain was not interested in promoting intellectual activity in this country, but there was some response to sporadic individual achievements. In any event, whatever the rulers thought, Indian society did not have a strong awareness of the importance of science, much less that of mathematics during the colonial days.

With the advent of independence, the national leadership – Jawaharlal Nehru in particular – laid great emphasis on science and propagated the idea of infusing our society with 'scientific temper'. Nehru's vision resulted in the creation of many institutions of scientific research and among them were a few that actively promoted mathematics. However even as there is a general perception of science as an important human activity, this perception is (understandably) based on the concrete and practical role science has had in industrial development. There is much less understanding of the civilizational role of fundamental science in general, of mathematics in particular: there is little appreciation of the fact that a great deal of today's applicable knowledge was at some period in the past basic science at its frontiers. This applies to mathematics, much more than to other sciences.

The glamour attached to physics, thanks to the developments in the field of nuclear energy and more recently to biology because of the recent discoveries in genetics helps attract public attention to these fields. Chemistry too has areas that catch public imagination. The Nobel Prize also helps increase public awareness of the importance of these sciences: the prize is a household word and some knowledge of facts about it is practically statutory requirement for school children taking part in quiz contests. Mathematics does not have the advantage of being able to project such glamorous images and Alfred Nobel unfortunately did not consider mathematics as worthy of partaking of his huge legacy. Most people, otherwise well-informed are not aware that there is something called the Fields Medal, which identifies superior mathematical achievement even as the Nobel does in other fields. Perhaps the fact that its monetary value is a pittance in comparison with the Nobel is the reason for this. Since its inception, about 50 mathematicians (necessarily under the age of forty) have been awarded the medal and if a school child happens to know the name of one of these recipients, it is a safe bet to assume that one of his/her parents is a mathematician. Even most of our teachers in schools and colleges are not aware of the existence of this medal for mathematics.

There is a general feeling that unlike physics or other sciences, mathematics is a somewhat other worldly pursuit. Few realize that esoteric problems of cosmology pursued by astrophysicists or the frontier areas of particle physics are as meaningful to the practical everyday world as Fermat's last theorem. As a mathematician I have come across many people who wonder what there was left in mathematics to discover, an experience that I am sure the physicist does not share.

Despite this general lack of public awareness about mathematics, there is in this country a vague feeling that we are very good at mathematics and there is a certain pride in its past achievements. No doubt we have major past contributions to mathematics to our credit; the invention of the place value system for representing numbers with the remarkable zero is undoubtedly one of the greatest achievements, a brilliant piece of abstract mathematics which is at the same time an indispensable practical tool in virtually every sphere of human activity. The romantic story of Srinivasa Ramanujan no doubt contributes (very justifiably) to this belief. But there are other dubious claims on behalf of vedic mathematics which are also taken quite seriously and contribute to this confidence in our mathematical proficiency. And there is this misconception that people like Shakuntala Devi who can perform calculating feats represent superior mathematical talent that strengthen this perception. Unfortunately, our track record in the twentieth century cannot quite justify such confidence. There have been some very substantial Indian contributions to the progress of mathematics since Ramanujan, but of much of this, the general public is not aware. Also while these have made the international community of mathematicians sit up and take notice of us, we cannot yet claim to be a leading force in world mathematics; what is worse, even as the peaks of our achievements have made their mark, a lot of what is passed off as original research in our institutions of higher learning is of a shamefully poor quality.

That then is briefly the situation of where India stands in relation to the higher reaches of mathematics. What of other levels? Let us begin with mathematics in schools. Already at this elementary level, mathematics seems to be viewed with a degree of uneasy fear not only by children but also by their parents. People, I assume, in general understand the importance of basic arithmetic in everyday life. There is however much less appreciation that an acquaintance with mathematics even at the elementary level, helps inculcate habits of thought that promote scientific temper.

Be that as it may, the main problem even at the elementary level is the paucity of competent teachers; and I am not talking of rural schools where even elementary infrastructure is not available – the lack of competence permeates the entire school system, the most elite being no exception. Lack of communication skills among our teachers is of course a contributory factor, but there is an emphasis on this aspect of the problem which results in our overlooking a much more serious dimension to it: the understanding of basic mathematics among many of these teachers is flawed, something that can be traced back to their own education. The small number of competent teachers are faced with enormous difficulties. They have to handle classes of inordinately large sizes. This is hard enough when all you have to do is to transmit information, but teaching mathematics, more than any other subject involves the transfer of ideas and concepts. Different children have different capacities for absorbing abstract ideas and often different ways of explaining things are needed to get them across to different children. Large classes inevitably make even very talented teachers ineffective and frustrated.

Syllabus reforms are a favourite preoccupation with our educational bodies. There is no doubt a need for periodic examination of what our children are taught and to modify them to keep in step with the times. But often enough, the bodies charged with this responsibility have not applied themselves adequately or have been wanting in other ways and have introduced ill-considered changes (copying mistakes done in the West). In any event reforms of this kind can make sense only in a context of competent human resources being available; and this is really a socio-economic problem quite outside the ambit of the educational bodies.

The school teacher in India today occupies a rather lowly place in our socio-economic ladder. The economic status has never been high, but in an earlier era the teacher was a highly respected member of the society which to some extent off-set the relative economic deprivation. But in an increasingly consumerist society, the reverence of that earlier era has all but disappeared and the economic status has also declined considerably. Not surprisingly, the teaching profession does not attract bright people any more and that is the real cause for the sorry state of our education at the school level.

Much of what I have said applies to the teaching profession at all levels; and teachers of all subjects at that. But mathematics has an added disadvantage *vis-à-vis* some other subjects like physics or chemistry or commerce: these latter are seen as potentially preparing you for a wide range of careers compared to mathematics. This results in fewer bright people specializing in mathematics than in many of these other subjects and the pool available to draw competent mathematics teachers (especially at levels beyond the school) from, is very small indeed.

The situation is reaching crisis proportions. With the decline in the quality of our teachers, less and less number of students emerge out of our education system with any proficiency in mathematics. And it is from out of this inadequately prepared lot that teachers and researchers of the next generation are emerging. If the trend is not reversed immediately we would have lost irretrievably such progress as we have made over this last century. The basic role that mathematics plays in every other field will ensure that the decline will affect every sphere of activity and the progress I am referring to is by no means confined to mathematics.

The need to reverse this trend will be readily admitted. However there has been no concrete initiative towards solving this immense problem. As I already said, the root causes are socio-economic. It is truly a sad situation that our society pays scant attention to the deplorable economic status of the teachers, the men and women who hold the future of coming generations in their hands. There ought to be some serious thinking about the kind of social engineering that would enable us to improve the place of our teachers in society. Without such steps it would appear that reforms in other aspects of our educational system are not likely to improve the situation except marginally.

Apart from improving the lot of our teachers, there are other steps that could improve the quality of students opting to study advanced mathematics and thereby contributing to the overall improvement in the situation. Indian industry and business are yet to recognize the role sophisticated mathematics has in modern technological developments as well as finance. They make little effort to promote any kind of research, leave alone mathematical research, relying rather on imported technologies. In the advanced countries, on the other hand mathematicians are hired by diverse industries while Wall Street is constantly on the look out for mathematics Ph Ds. Should we follow this Western lead, there would be many more avenues of work open for the mathematically trained and that will certainly encourage the mathematically talented to pursue the subject.

The one silver lining in the cloud is that despite all the odds stacked against it, there are young men and women – admittedly a very small number, and getting even smaller – who doggedly pursue mathematics and are performing at superior international levels.