On generation of magnetic field in astrophysical bodies

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In this paper we compare energy transfer rates in magnetohydrodynamic turbulence using field-theoretic method. Our calculation shows that there is a large energy transfer rate from the large-scale velocity field to the large-scale magnetic field. The growth of large-scale magnetic energy is primarily due to this transfer. We reached the above conclusion without any linear approximation like that in α-dynamo.

THE generation of magnetic field in astrophysical bodies, e.g. galaxies, stars and planets is one of the outstanding theoretical problems of physics and astrophysics. The initial magnetic fields of galaxies and stars are weak, and are amplified by the turbulent motion of the plasma. The generated field gets saturated due to nonlinear interactions. The above process is called 'dynamo' action. Qualitatively, the magnetic field is amplified by the stretching of the field lines due to turbulent plasma motion. A fraction of kinetic energy of the plasma is spent in increasing the tension of the magnetic field lines, which effectively enhances the magnetic field strength. Current dynamo theories of are of two types, kinematic and dynamic. In the kinematic theories, one studies the evolution of magnetic field under a prescribed velocity field. In kinematic α-dynamo, the averaged nonlinear term $\langle \mathbf{u} \times \mathbf{b} \rangle$ ($\mathbf{u}$, $\mathbf{b}$ are velocity and magnetic field fluctuations respectively) is replaced by a constant α times mean magnetic field $\mathbf{B}_0$. This process, which is valid for small magnetic field fluctuations, yields linear equations that can be solved for a given boundary condition and external forcing fields[1–3]. In dynamic theories[4–6], the modification of velocity field by the magnetic field (back reaction) is taken into account. Using a different approach, here we compute energy transfer rates from velocity field to magnetic field using field-theoretic method. The striking result of our field theoretic calculation is that there is a large energy transfer rate from the large-scale velocity field to the large-scale magnetic field. We claim that the growth of large-scale magnetic energy is primarily due to this transfer.

There is an exchange of energy between various Fourier modes because of nonlinear interactions present in magnetohydrodynamics (MHD). Since there are two vector fields $\mathbf{u}$ and $\mathbf{b}$ in MHD, the energy can be transferred from $\mathbf{u}$ to $\mathbf{u}$, $\mathbf{u}$ to $\mathbf{b}$, and $\mathbf{b}$ to $\mathbf{b}$. Energy from a parent mode $\mathbf{u}(\mathbf{k})$ or $\mathbf{b}(\mathbf{k})$ ($\mathbf{k}$ represents the wave number of Fourier mode) is transferred to two daughter modes with wave numbers $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$. The allowed triads in MHD are ($\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{p})$, $\mathbf{u}(\mathbf{k}-\mathbf{p})$) and ($\mathbf{u}(\mathbf{k})$, $\mathbf{b}(\mathbf{p})$, $\mathbf{b}(\mathbf{k}-\mathbf{p})$). The net effects of all the energy transfers are constant energy fluxes from large-scale u to small-scale u ($\Pi^{u<}_{u>}$), large-scale u to small-scale b ($\Pi^{u<}_{b>}$), large-scale b to small-scale u ($\Pi^{b<}_{u>}$), large-scale b to small-scale b ($\Pi^{b<}_{b>}$), and large-scale u to large-scale b ($\Pi^{u<}_{b<}$). The superscript and subscript of Π refer to the source and sink respectively. All these energy fluxes are illustrated in Figure 1. These energy fluxes are analogous to that of Kolmogorov's flux in fluid turbulence. Note that large-scale velocity modes are forced, as shown in Figure 1.

Stanišić[7], Dar *et al.*[8], and others have given formulae for computation of the above fluxes. However, the formalism by Dar *et al.*[8] is the most general; they have numerically computed all the fluxes of MHD using numerical data of direct numerical simulation. In the present paper, we compute the MHD fluxes in the inertial range using field-theoretic method. Our calculation is up to first-order in perturbation.

We give a brief outline of the theoretical calculation (refer to Verma[9] for details). We write down the evolution equation for kinetic energy spectrum ($\langle |\mathbf{u}(\mathbf{k})|^2 \rangle/2$) and magnetic energy ($\langle |\mathbf{B}(\mathbf{k})|^2 \rangle/2$). We carry out our analysis for three space dimensions, constant mass density, and zero mean magnetic field. We also assume that $\langle \mathbf{u}\cdot\mathbf{b} \rangle = 0$, and that the large-scale velocity modes are forced. There is a net outflow/inflow of energy from a wave number

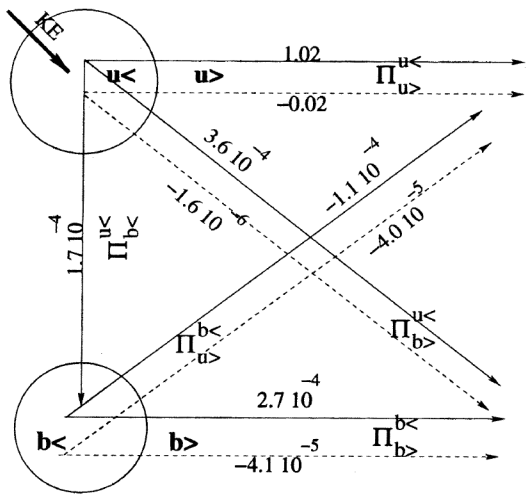

Figure 1. Various energy fluxes in wave number space for parameter ($r_A = 5000$, $r_K = 0.1$, $r_M = -0.1$). The illustrated wave number spheres contain $\mathbf{u}<$ and $\mathbf{b}<$ modes, while $\mathbf{u}>$ and $\mathbf{b}>$ are modes outside these spheres. The energy fluxes are from large-scale u to small-scale u ($\Pi^{u<}_{u>}$), large-scale u to small-scale b ($\Pi^{u<}_{b>}$), large-scale b to small-scale u ($\Pi^{b<}_{u>}$), large-scale b to small-scale b ($\Pi^{b<}_{b>}$), and large-scale u to large-scale b ($\Pi^{u<}_{b<}$). The velocity fields at large-scale are forced, and the net input energy is 1 unit. Solid and dashed lines represent the nonhelical and helical contributions respectively.

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sphere (say sphere of radius k_0) as discussed earlier. These energy fluxes can be easily calculated using the Fourier modes. To illustrate, the energy flux from the modes inside of the u-sphere of radius k_0 to the modes outside of the b-sphere of radius k_0 is given by

$$\Pi_{b>}^{u<}(k_0) = \int_{k'>k_0} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \int_{p>k_0} \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \times \langle \Im([\mathbf{k}\cdot\mathbf{u}(\mathbf{q})][\mathbf{b}(\mathbf{k})\cdot\mathbf{u}(\mathbf{p})])\rangle, \quad (1)$$

where $\Im$ stands for the imaginary part, and $\mathbf{k} = \mathbf{p} + \mathbf{q}$. Clearly, the sources of energy for the $b<$ sphere are $\Pi_{b<}^{u<}$, $\Pi_{b<}^{u>}$ and $\Pi_{b<}^{b>}$. If there is a net flux of energy into large-scale magnetic energy, then magnetic energy at large-scale will grow, or dynamo will be active.

We calculate the energy fluxes perturbatively to first order. We assume homogeneity and isotropy for the flow. In the correlation functions we have included kinetic helicity ($H_K = \langle \mathbf{u}\cdot\omega\rangle/2$, where ω is the vorticity) and magnetic helicity ($H_M = \langle \mathbf{a}\cdot\mathbf{b}\rangle/2$, where $\mathbf{a}$ is the vector potential), which are defined using

$$\langle u_i(\mathbf{k}) u_j(\mathbf{k}')\rangle = \left[P_{ij}(\mathbf{k}) C^{uu}(\mathbf{k}) - i\varepsilon_{ijl} k_l \frac{2H_K(k)}{k^2}\right]\delta(\mathbf{k}+\mathbf{k}'), \quad (2)$$

$$\langle b_i(\mathbf{k}) b_j(\mathbf{k}')\rangle = [P_{ij}(\mathbf{k}) C^{bb}(\mathbf{k}) - i\varepsilon_{ijl} k_l H_M(k)]\delta(\mathbf{k}+\mathbf{k}'). \quad (3)$$

Note that helicities break mirror symmetry.

We focus on the fluxes in the inertial range. On the basis of recent theoretical[10–14] and numerical[15,16] evidences, we take Kolmogorov's spectrum for the correlation functions in the inertial range, i.e.

$$C^{uu}(\mathbf{k}) = \frac{K^u}{4\pi}\Pi^{2/3}k^{-11/3}; \; C^{bb}(\mathbf{k}) = C^{uu}(\mathbf{k})/r_A; \quad (4)$$

$$H_K(\mathbf{k}) = r_K k C^{uu}(\mathbf{k}); \; H_M(\mathbf{k}) = r_M C^{bb}(\mathbf{k})/k, \quad (5)$$

where Π is the total cascade rate, and K^u is the Kolmogorov's constant.

Since both magnetic and kinetic energy spectrum are Kolmogorov-like in the inertial range, r_A, r_K and r_M can be treated as constants. We also use turbulent or renormalized viscosity and resistivity in our calculation. These quantities have been recently derived by Verma[13,17]. Using the steady-state condition, we also calculate the energy supply from the large-scale velocity field to the large-scale magnetic field $\Pi_{b<}^{u<} = \Pi_{b>}^{b<} + \Pi_{u>}^{b<}$. Since the inertial-range energy spectrum is universal, the above energy flux is independent of the details of large-scale forcing.

Figure 1 shows both nonhelical (solid line) and helical (dashed line) contributions for the case ($r_A = 5000$, $r_K = 0.1$, $r_M = -0.1$). Large r_A (here $r_A = 5000$) corresponds to a very weak magnetic field compared to the velocity field, similar to the early phase of galactic and stellar evolution. The choice of $r_K = 0.1$ and $r_M = -0.1$ is motivated by the fact that both kinetic and magnetic helicities in most of the astrophysical plasmas are relatively small. The recent simulations of Brandenburg[18] as well as the EDQNM calculation of Pouquet *et al.*[4] show that the magnetic helicity is negative for small wave numbers. Since the small wave number contributions dominate those from the large wave numbers, we have taken $r_M < 0$.

The flux ratios shown in Figure 1 illustrate some important results. They are:

(i) There is a large energy flux from large-scale velocity field to large-scale magnetic field ($\Pi_{b<}^{u<}$). In addition, there are two other fluxes, $\Pi_{b<\mathrm{helical}}^{b>} + \Pi_{b<\mathrm{helical}}^{u>}$, to the large-scale magnetic field. These fluxes are responsible for the growth of large-scale magnetic field in the initial stage of evolution. Pouquet *et al.*[4], Pouquet and Patterson[19], Brandenburg[18], and others generally highlight $\Pi_{b<\mathrm{helical}}^{b>}$ inverse transfer, and do not consider $\Pi_{b<}^{u<}$. Recently, Brandenburg[18,20] had argued that large-scale magnetic energy is sustained by nonlocal inverse cascade from the forcing scale directly to the largest scale of the box. He relates this effect to the α-effect. In this paper and in ref. 9, we have computed the relative magnitudes of all three contributions for generic parameters discussed above, and find all of them to be comparable; however, $\Pi_{b<}^{u<}$ is somewhat higher.

(ii) As shown in Figure 1 the nonhelical component of magnetic energy flux ($\Pi_{b>}^{b<}$) is forward, while the helical component of magnetic energy flux is negative (inverse). The overall magnetic energy flux, however, is positive.

In our theoretical calculation we have assumed homogeneity and isotropy of the flow, as well as $\mathbf{B}_0 = 0$. These assumptions are likely to hold in the early stages of the galactic evolution before large structures appear. The galaxies are known to be disc-shaped (two-dimensional) and typically of the spiral form. Hence, our theoretical results may seem inapplicable in the case of galaxies. However, we believe that physics described here is relevant at the intermediate scales in the spirit of Kolmogorov's universal theory of turbulence. The energy transfer at the intermediate scales is independent of large-scale structures and small-scale dissipation processes. In addition, MHD turbulence exhibits similar cascade behaviour in both two and three dimensions[8,13]. Hence, our model can be an important ingredient in the full-fledge galactic model even though it cannot by itself describe the evolution of galactic magnetic field. We assume that the energy cascades in galaxies can be

described by the model presented here, and attempt to estimate the magnetic energy growth time-scale.

To model the early evolution of galaxies, we assume that the large-scales contain kinetic and magnetic energies. During this unsteady period, magnetic energy (E^b) will be amplified by the nonhelical $\Pi_{u<}^{b<}$, and helical components $\Pi_{b<\text{helical}}^{b>}$ and $\Pi_{b<\text{helical}}^{u>}$, i.e.

$$\frac{\mathrm{d}E^b(t)}{\mathrm{d}t} = \Pi_{b<}^{u<} + \Pi_{b<\text{helical}}^{b>} + \Pi_{b<\text{helical}}^{u>}. \tag{6}$$

We assume quasi-steady state for the galactic evolution. Our theoretical calculation performed for large r_A shows that the energy fluxes on the right-hand-side of eq. (6) are proportional to $\Pi E^b/E^u$, where E^u is the kinetic energy, and Π is the total energy flux or energy supply rate. Using Kolmogorov's spectrum, we obtain

$$E^b(t) \approx E^b(0) \exp\left(\frac{\sqrt{E^u}}{L(K^u)^{3/2}}\, t\right), \tag{7}$$

where L is the large-length of the system and $K^u \approx 1$. Clearly, the magnetic energy grows exponentially in the early periods and the time-scale of growth is of the order of $L/(E^u)^{1/2}$, which is the eddy turnover time. Taking $L \approx 10^7$ km and $(E^u)^{1/2} \approx 10$ km/s, we obtain the growth time-scale to be 10^{16} s or 3×10^8 years, which is in the expected range[21]. Hence, we are able to construct a nonlinear and dynamically consistent galactic dynamo based on the energy fluxes. In this model, magnetic energy grows exponentially, and the growth time-scale is reasonable[21]. To recapitulate, we address the dynamo problem in the light of turbulent energy flux.

Our approach is different from the α-dynamo picture. Our theoretical calculation, based on perturbative field theory, shows large amount of energy transfer from large-scale velocity field to large-scale magnetic field. This transfer occurs in both helical and nonhelical MHD. We believe magnetic energy growth is primarily due to this energy transfer. Regarding magnetic energy flux, there is (i) nonhelical forward flux from large scales to small scales, and (ii) helical inverse transfer from small scales to large scales. The net magnetic energy transfer, however, is positive.

We have constructed a model of galactic dynamo based on our energy flux results. We find that magnetic energy grows exponentially, and our estimate of its growth time-scale is consistent with the current observational estimates[21].

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