Option pricing: An overview

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A brief introduction to stock options and the theory of its pricing is given. Important concepts like hedging strategies, complete markets are explained via an illustrative example. These notions are formalized in the setting of a discrete model of underlying stock prices.

THE topic of option pricing has attracted a lot of attention internationally since Scholes and Merton were awarded the Nobel Prize for Economics in 1997. It has been in news in India recently as option trading has been allowed in Indian markets and the trading in options is about to begin in Indian stock markets. In this article, I will attempt to give an introduction to options, the theory of its pricing and its interplay with probability theory. The important ideas will be presented via an example and then these will be formalized in the context of a discrete model. I will conclude with a discussion on continuous time models. For proofs of results stated here as well as further discussion, historical perspective and other references, one can refer to Kallianpur and Karandikar[1].

European call option

We will first discuss European call options. A European call option is a one-way contract between the seller (of option) and the buyer – it specifies shares of a specific company, a terminal time T, a strike price K and it entitles the buyer to buy one share of the specified company at the terminal time T at the strike price K, irrespective of the prevailing price in the market at that time.

The option entitles the buyer to buy the share, but there is no obligation to buy. On the other hand, the seller of option is obliged to sell if the buyer so desires. It is clear that buyer would exercise the option if the market price is above K, while the buyer would not exercise the option if the market price is less than K.

What should be the price of the option? Obviously, the answer would depend upon the available information about the shares of the company in question, which should be translated into a mathematical model for the share prices. Obviously, the only realistic mathematical model that can be considered is a stochastic model.

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We will be considering an ideal situation with two simplifying assumptions: (1) There are no transaction costs (in buying or selling shares); (2) The rate of interest on investments is same as that on loans. These two assumptions are not true is practice, but are approximately correct when one is considering transactions between two large brokers/mutual funds.

Example

We will now consider a concrete example and use it to illustrate the ideas that play an important role in option pricing. This is an artificial example. Its role is only to explain the notions such as no arbitrage, hedging strategy, complete markets, etc.

To avoid technicalities, we will consider a discrete model. We will consider a company whose shares are trading at the initial time ($t = 0$) @ S_0 (in rupees) per share. We assume that no trading is allowed in the share for a period of one year, at the end of which the price is S_1. Again, no trading is allowed for another year when the price becomes S_2.

Let the stochastic model for (S_0, S_1, S_2) be given by

$$P(S_0 = 4000) = 1, \quad (1)$$

$$P(S_1 = 4950) = 0.5, \quad (2)$$

$$P(S_1 = 3850) = 0.5, \quad (3)$$

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.1, \quad (4)$$

$$P(S_2 = 8470 \mid S_1 = 4950) = 0.4, \quad (5)$$

$$P(S_2 = 3630 \mid S_1 = 4950) = 0.5, \quad (6)$$

$$P(S_2 = 6655 \mid S_1 = 3850) = 0.5, \quad (7)$$

$$P(S_2 = 3630 \mid S_1 = 3850) = 0.5. \quad (8)$$

Let us assume that the common rate of interest for loans as well as deposits is 10% per year.

Suppose that also selling in the market is European call option on these shares, with terminal time $T = 2$ years, stike price $K = 6050$. At what price should this option be traded in a market in equilibrium (which means enough buyers and sellers will be there in the market at this price)?

At a first glance it would appear, at least to readers familiar with probability theory, that the option price must be the expected return. In this case, if at the end of two years, S_2 is more than 6050, the gain is ($S_2 - 6050$) (an investor who has bought the option can buy a share @

Rs 6050 and sell it at S_2); whereas if $S_2 < 6050$, the gain is 0 (the buyer of the option need not buy the share at all). Thus the gain is

$$\max(S_2 - 6050, 0).$$

This gain is due at the end of 2 years. Its worth at time zero (with rate of interest 10%) is

$$\max(S_2 - 6050, 0)/1.21.$$

Thus the expected gain is

$$g = E(\max(S_2 - 6050, 0)/1.21).$$

Here, $P(S_2 = 9680) = 0.05$, $P(S_2 = 8470) = 0.2$, $P(S_2 = 6655) = 0.25$, $P(S_2 = 3630) = 0.5$. This leads to $g = 675$. Can the price of the option be Rs 675? Suppose that options are trading @ Rs 675.

An investor A decides to buy 100 options by investing Rs 67500. Another investor B also decides to invest Rs 67500 ($= x_0$) at time 0; buy $\pi_0 = 75$ shares @ Rs 4000 by borrowing the shortfall. At the end of the year, if the price is $S_1 = 4950$, he sells 10 shares to bring down his holding to $\pi_{11} = 65$, using the proceeds to settle part of this loan. If $S_1 = 3850$, he sells 50 shares to bring his holding to $\pi_{12} = 25$, again paying off loan with the money received. Denoting the net deposit by the investor at time 0 by ξ_0 and at time 1 by ξ_{11} if $S_1 = 4950$; ξ_{12} if $S_1 = 3850$ (negative ξ means loan), ξs are determined by

$$\xi_0 = x_0 - \pi \times 4000, \tag{9}$$

$$\xi_{11} = (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11}, \tag{10}$$

$$\xi_{12} = (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}. \tag{11}$$

For the investor B, $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$ and eqs (9)–(11) give $\xi_0 = -232500$, $\xi_{11} = -206250$, $\xi_{12} = -63250$.

Before proceeding further, let us note that a trading strategy is determined by x_0, π_0, π_{11}, π_{12} which in turn determine ξ_0, ξ_{11}, ξ_{12}.

Table 1 shows the net worth of the holdings of A, B in each of the five possible outcomes of (S_1, S_2): (A's assets are 100 options and no liabilities; B's assets are π_{11} (π_{12}) shares and a deposit of ξ_{11} (ξ_{12}) made at time 1 if $S_1 = 4950$ ($S_1 = 3850$)).

Note that while both A and B made the same initial investment, namely Rs 67500, B has done better than A in each possible outcome of the stock prices. So whatever A was buying is overpriced. Thus option price must be less than Rs 675!

Table 1. Net worth of the holdings

Outcome (S_1, S_2)	A's holding (Rs)	B's holding (Rs)
(4950, 9680)	363000	402325
(4950, 8470)	242000	323675
(4950, 3630)	0	9075
(3850, 6655)	60500	96800
(3850, 3630)	0	21175

To see this more clearly, assume that options are priced at Rs 675 and there are enough buyers and sellers at this price.

An investor C devises a strategy as follows: Sell 100 options @ Rs 675 per option to collect Rs 67500 and then follow the strategy of B: $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$.

Then the net worth of C's holding is given by the difference of the 3rd column and 2nd column in Table 1 (see Table 2).

Thus, C would make a profit in each of the five outcomes without making any initial investment. Clearly, every investor would like to follow this strategy and make money without taking any risk. This in turn would disturb the equilibrium and soon there would be no buyers for the option.

The strategy of C is an example of an Arbitrage opportunity.

Arbitrage opportunity is a strategy that involves no initial investment and for which the net worth of holdings (at some time in future) is non-negative for each possible outcome and strictly positive for at least one possible outcome.

As explained above, if an Arbitrage opportunity exists, it would disturb the equilibrium, as all investors would like to replicate the same. Thus, in a market in equilibrium, Arbitrage opportunities do not exist. This is known as the principle of no arbitrage abbreviated as NA. In this example we can conclude

$$\text{NA} \Rightarrow p < 675,$$

(where p is the price of the option).

Now, let us consider another investor D's strategy: $x_0 = 30000$, $\pi_0 = 50$, $\pi_{11} = 45$, $\pi_{12} = 15$. Equations (9)–(11) yield: $\xi_0 = -170000$, $\xi_{11} = -162250$ $\xi_{12} = -52250$. In each of the outcomes, the net worth of D's holding is given in Table 3.

Table 2. Net worth of C's holding

Outcome (S_1, S_2)	C's holding (Rs)
(4950, 9680)	39325
(4950, 8470)	81675
(4950, 3630)	9075
(3850, 6655)	26300
(3850, 3630)	21175

Table 3. Net worth of D's holding

Outcome (S_1, S_2)	D's holding
(4950, 9680)	$9680\pi_{11} + 1.1\xi_{11} = 257125$
(4950, 8470)	$8470\pi_{11} + 1.1\xi_{11} = 202675$
(4950, 3630)	$3630\pi_{11} + 1.1\xi_{11} = 15125$
(3850, 6655)	$6655\pi_{11} + 1.1\xi_{12} = 42350$
(3850, 3630)	$3630\pi_{11} + 1.1\xi_{12} = -3025$

Note that for each outcome, D's holdings are worth less than 100 options. As a consequence, the price of 100 options is more than the investment at time zero that is needed for D's strategy, namely 30000. Thus, one option is worth more than 300.

If indeed option price is Rs 300 or less, then the strategy consisting of buying 100 options and $\pi_0 = -50\pi_{11} = -45$, $\pi_{12} = -15$ would be an arbitrage opportunity (note that the πs are (-1) times the corresponding π in D's strategy. Thus

$$\text{NA} \Rightarrow p > 300.$$

Can we narrow the interval (300, 675) for the option price any further?

It is clear from the preceding discussion that if for a trading strategy $(x_0, \pi_0, \pi_{11}, \pi_{12})$ (with $\xi_0, \xi_{11}, \xi_{12}$ given by eqs (9)–(11)

$$9680\pi_{11} + 1.1\xi_{11} \geq 363000, \quad (12)$$

$$8470\pi_{11} + 1.1\xi_{11} \geq 242000, \quad (13)$$

$$3630\pi_{11} + 1.1\xi_{11} \geq 0, \quad (14)$$

$$6655\pi_{12} + 1.1\xi_{12} \geq 60500, \quad (15)$$

$$3630\pi_{12} + 1.1\xi_{12} \geq 0, \quad (16)$$

then $100p \leq x_0$ and if any of the eqs (12)–(16) is a strict inequality, $p < x_0$. Likewise, if $(x_0, \pi_0, \pi_{11}, \pi_{12})$, $(\xi_0, \xi_{11}, \xi_{12})$ satisfy eqs (9)–(11) and

$$9680\pi_{11} + 1.1\xi_{11} \leq 363000, \quad (17)$$

$$8470\pi_{11} + 1.1\xi_{11} \leq 242000, \quad (18)$$

$$3630\pi_{11} + 1.1\xi_{11} \leq 0, \quad (19)$$

$$6655\pi_{12} + 1.1\xi_{12} \leq 60500, \quad (20)$$

$$3630\pi_{12} + 1.1\xi_{12} \leq 0, \quad (21)$$

then $x_0 \leq p$ and if any of eqs (17)–(21) is a strict inequality, $x_0 < 100p$.

Thus, the optimum value x^+ for the linear programming problem (I)

minimize x_0
subject to eqs (9)–(16)

is an upper bound for $100p$; and if for the optimum solution, even one of the eqs (12)–(16) is strict inequality, then $100p < x^+$.

The optimum value x^- for the linear programming problem (II)

maximize x_0
subject to eqs (9)–(11) and eqs (17)–(21)

is a lower bound for p; and if for the optimum solution, even one of the eqs (17)–(21) is strict inequality, then $x^- < 100p$.

The optimum solution to the problem (I) is $x_0 = 50000$, $\pi_0 = 80$, $\pi_{11} = 60$, $\pi_{12} = 20$, $\xi_0 = -270000$, $\xi_{11} = -198000$, $\xi_{12} = -66000$ with eq. (13) being a strict inequality. Thus $100p < 50000$.

The optimum solution to the problem (II) is $x_0 = 42500$, $\pi_0 = 65$, $\pi_{11} = 50$, $\pi_{12} = -66000$ with eq. (17) being a strict inequality. Thus $100p > 42500$, hence we conclude that

$$425 < p < 500.$$

Let us explore an alternate scenario. Suppose that instead of eqs (4) and (5) one has

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.5.$$

In this case, the upper bound x^+ is solution to the problem III:

minimize x_0
subject to eqs (9)–(12) and (14)–(16).

In this case the optimum is attained by the same strategy that optimized problem I, with $x_0 = 50,000$. The lower bound x^- is given by solution to problem IV

maximize x_0
subject to eqs (9)–(11), eq. (17) and eqs (19)–(21).

Again, the optimum solution is the same as the one for problem III with $x_0 = 50,000$.

Thus $x^- = 50000$, $x^+ = 50000$. It follows that the option price must be 500.

In both problems III and IV, all constrains are equalities for the optimum solution. Thus, with initial investment $x_0 = 50000$, there exists a strategy: $\pi_0 = 80$, $\pi_{11} = 60$ and $\pi_{12} = 20$ for which the net worth of the holdings at the end of 2 years is exactly the same as the worth of 100 options for all possible outcomes of the share prices. Such a strategy is called a hedging strategy for the options.

Likewise, if instead of eqs (4) and (5) one has

$$P(S_2 = 8470 \mid S_1 = 4950) = 0.5, \quad (22)$$

then again, the upper and lower bounds agree and a hedging strategy exists: the strategy that was solution to problem II with $x_0 = 42500$. Thus in this case, the option price is $p = 425$.

Note that in both these cases where a hedging strategy exists, the option price did not depend upon the probabilities of the various outcomes, but it depended upon the set of possible outcomes. This is so because we are matching the returns for each outcome and so it does not matter as to with what probability an outcome occurs Thus expected value of the (discounted) gain can be more or less than the option price.

So what went wrong with the reasoning *price = expected gain*? The reason is that along with the option, another commodity, namely the shares of the same company, is also available in the market and of course, the shares are correlated with the option – and thus we need to valuate the option in terms of a basket consisting of money and shares.

If the shares of the company were not being traded but only the options were being sold, then perhaps the expected (discounted) gain can be taken as the price (if the utility is taken as linear).

Let us return to the example. Now suppose that eqs (4) and (6) are replaced by

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.6, \tag{23}$$

(the main point being, if $S_1 = 4950$, there are only two possibilities: $S_2 = 9680$ or $S_2 = 8470$). In this case, the upper bound x^+ is solution to problem V:

minimize x_0
subject to eqs (9)–(13), eqs (15) and (16),

and it can be seen that $x^+ = -\infty$. The lower bound x^- is solution to problem VI:

maximize x_0
subject to eqs (9)–(11), eqs (17) and (18), eqs (20) and (21),

and here $x^- = \infty$. Thus no price is feasible, i.e. irrespective of the price of the option, there will be arbitrage opportunities. The reason is that arbitrage opportunities exist even without bringing in options. The strategy $x_0 = 0$, $\pi_0 = 0$, $\pi_{11} = 100$, $\pi_{12} = 0$ is an arbitrage opportunity! Thus, if we are to adopt NA as a basis for pricing options, we must rule out options in the market consisting only of shares. How do we do this?

We will now examine these issues and questions such as: Do hedging strategies exist for the call option and the put option?

General discrete model

In this section, we will consider a discrete model for the share price of a company and discuss arbitrage opportunities, hedging strategies and option pricing.

To keep technicalities to a minimum, we assume that trading takes place at multiples of a fixed time period, say days; that we are considering a finite horizon – N days and that the share price S_k on the kth day is a random variable that takes only finitely many values (with S_0 being a constant).

Without loss of generality, we can (and do) assume that the random variables are defined on the probability space Ω consisting of all possible outcomes of ($S_0, S_1, \ldots . S_k, \ldots, S_N$). Thus, Ω is a finite subset of $[0, \infty)^{N+1}$. The points w of Ω are denoted by

$$w = (s_0, s_1, \ldots, s_N).$$

The probability P on Ω is a function from $\Omega \to [0, 1]$ such that $P(\{w\}) > 0$ for all $w \in \Omega$ (as Ω consists of all *possible* outcomes). The random variables S_k are defined by

$$S_k(w) = s_k, \text{ where } w = (s_0, \ldots, s_N).$$

As we saw in the example, it is possible that a model for share prices may admit arbitrage opportunities. We must rule out such models. To do this, we need to define what an *arbitrage opportunity* is and for that we need to define a *trading strategy*.

As in the example, we will only consider self-financing trading strategies – where the only investment is at time zero and at any subsequent time, there is no investment or consumption.

We will assume that we are considering frictionless market: (i) there are no transaction costs, and (ii) the rate of interest on loans is the same as that on investments, say r per day.

A (self-financing) trading strategy is described by initial investment z and ($\pi_0, \ldots, \pi_{N-1}$), where π_k denotes the number of shares the investor decides to posses on kth day. If $\pi_{k-1} < \pi_k$, he/she buys ($\pi_k - \pi_{k-1}$), while if $\pi_k < \pi_{k-1}$, he/she sells ($\pi_{k-1} - \pi_k$) shares. In the first case, he/she borrows an additional amount $(\pi_k - \pi_{k-1})S_k$ and in the second case investing the amount realized from the sale: $(\pi_{k-1} - \pi_k)S_k$.

Clearly, an investor can choose $\pi_0, \ldots, \pi_k, \ldots, \pi_{N-1}$. On kth day while deciding upon π_k, he/she can use the information available then, namely the history of the share prices till then: $S_0, \ldots, S_k$. π_k cannot be allowed to depend on $S_{k+1}, \ldots, S_N$: A strategy such as $\pi_k = 1000$, if $S_{k+1} < S_k$ and $\pi_k = 0$, if $S_{k+1} \geq S_k$ is not admissible as the investor on kth day does not know if the next day, the share price will go up or down. So π_k should be a function of ($S_0, \ldots S_k$). To make this precise, for $0 \leq k \leq N$, let

$$H_k\{(s_0, s_1, \ldots, s_k): P(S_0 = s_0, S_1 = s_1, \ldots, S_k = s_k) > 0\}.$$

A trading strategy is (z, π), where z is the initial investment, $\pi = (\pi_0, \ldots, \pi_{N-1})$, π_k is a function from H_k into $\mathbb{R}$.

Note that since the strategy is self-financing, the investment (or loan) ξ_k at time k can be determined from $x, \pi_0, \ldots, \pi_k, S_0, \ldots, S_k$.

With a little bit of algebra, it can be shown that worth V_k of the holdings of the investor following the strategy (x, π) on the kth day (before buying or selling on kth day) is given by

$$V_k(x, \pi)(w) = (1+r)^k \left\{ x + \sum_{j=0}^{k-1} \pi_j(S_0(w), \ldots, S_j(w)) \times \left(\frac{S_{j+1}(w)}{(1+r)^{j+1}} - \frac{S_j(w)}{(1+r)^j} \right) \right\}.$$

It is convenient to express share prices and value in terms of discounted prices: Thus let $\tilde{V}_k(x, \pi) = V_k(x, \pi)(1+r)^{-k}$, $\tilde{S}_k = S_k(1+r)^{-k}$. Then one has

$$\tilde{V}_k(x,\pi)(w) = x + \sum_{j=0}^{k-1} \pi_j(S_0(w),\ldots,S_j(w) \times \{\tilde{S}_{j+1}(w) - \tilde{S}_j(w)\}. \quad (24)$$

Definition: An arbitrage opportunity (in the market consisting of shares of this company) is a trading strategy $(0, \pi)$ such that

$$\tilde{V}_N(0,\pi)(w) \geq 0 \quad \forall\, w \in \Omega, \quad (25)$$

and

$$\tilde{V}_N(0,\pi)(w^*) > 0 \quad \text{for some } w^* \in \Omega. \quad (26)$$

The following result gives a necessary and sufficient condition for no arbitrage. For its proof see [1].

Theorem: The following are equivalent:
(i) There does not exist an arbitrage opportunity (in the market consisting of shares of this company);
(ii) For each k, $0 \leq k < N$; $(s_0, \ldots, s_k) \in H_k$.

$$P(S_{K+1} \geq (1+r)S_k \mid S_0 = s_0, \ldots, S_k = s_k) = 1$$
$$\Rightarrow P(S_{K+1} = (1+r)\,S_k, \mid S_0 = s_0, \ldots, S_k = s_k) = 1,$$

and

$$P(S_{K+1} \leq (1+r)S_k \mid S_0 = s_0, \ldots, S_k = s_k) = 1$$
$$\Rightarrow P(S_{K+1} = (1+r)\,S_k, \mid S_0 = s_0, \ldots, S_k = s_k) = 1.$$

If the share price model $\{S_k\}$ satisfies (i) in the theorem above, then we say that model satisfies NA.

There is an alternate characterization of NA, via Martingales. Recall that a process $\{Z_k = k \geq 0\}$ is said to be a Martingale if $E(Z_{K+1} \mid Z_0, \ldots, Z_k) = Z_k$ for all k.

Theorem: The share price model satisfies NA if and only if there exists a probability measure Q on Ω such that $Q(\{w\}) > 0$ for all $w \in \Omega$ and $\{S_k\colon 0 \leq k \leq N\}$ is a Martingale on (Ω, Q).

A measure Q as in the theorem stated above if it exists, is called an Equivalent Martingale Measure (EMM).

In the rest of the section, we assume that NA is true. We also fix an EMM Q.

We now come to the question of pricing of an option. Consider a European call option with terminal time N and strike price K. Let A^+ be the set of x such that there exists a strategy π satisfying

$$V_N(x,\pi)(w) \geq (S_N(w) - K)^+ \quad \forall\, w \in \Omega. \quad (27)$$

The right side of eq. (27) represents the gain from an option. Thus if (x, π) satisfies eq. (27), the option cannot be priced at a price above x (for then the strategy: sell one option and then follow the strategy π would be an arbitrage opportunity in the market consisting of the shares and the option.) Thus

$$x^+ = \inf A^+$$

is an upper bound for the price p of the option. Similarly, $x^- = \sup A^-$ is a lower bound for p, where A^- is the set of all y such that there exists a strategy π satisfying

$$V_N(y,\pi)(w) \leq (S_N(w) - K)^+ \quad \forall\, w \in \Omega. \quad (28)$$

Standard results from Martingale theory imply that if $\tilde{S}_k$ is a Martingale, then $\tilde{V}_k$ is a Q-Martingale and so

$$E_Q(\tilde{V}_k(x,\pi)) = x. \quad (29)$$

This implies

$$x^- \leq E_Q \frac{(S_N - K)^+}{(1+r)^N} \leq x^+, \quad (30)$$

showing that the bounds x^- and x^+ are consistent (provided NA holds).

A strategy (x, π) is said to be a hedging strategy for the option (under consideration) if

$$V_N(x,\pi)(w) = (S_N(w) - K)^+ \text{ for all } w \in \Omega.$$

In this case, $x \in A^+$ and $x \in A^-$ and so $x \geq x^+$ and $x \leq x^-$. This and eq. (30) give

$$x = x^- = x^+ = E_Q\left(\frac{(S_N - K)^+}{(1+r)^N}\right).$$

Thus, if a hedging strategy exists, the price is the initial investment required in the hedging strategy. The price also equals the expected discounted return, where the expectation is taken with respect to an EMM.

Let $g\colon \Omega \to [0, \infty]$. A broker is selling a contract that pays an amount $g(S_0, \ldots, S_N)$ at time N, where $S_0, \ldots, S_N$ are observed values of share prices. Such a contract is called a contingency claim.

When $g(w) = (S_N(w) - K)^+$, it is the European call option and when $g(w) = (K - S_N(w))^+$, it is the European put option. It entitles the buyer of the put option to sell one share at price K at time N. Let us briefly discuss as to how to price such a claim without introducing arbitrage opportunities.

Let $A^+(g)$ be defined as in the case of call option, with $g(w)$ in place of $(S_N(w) - K)^+$ in the right hand sides of eqs (27) and (28) and let $x^+(g) = \inf A^+(g)$, $x^-(g) = \sup A^-(g)$. There are the upper and lower bounds for the price of the contingent claim g. One also has

$$x^-(g) \leq E_Q(g(w))\,(1+r)^{-N} \leq x^+(g).$$

A strategy (x, π) is said to be a hedging strategy for g, if

$$V_N(x,\pi)(w) = g(w) \quad \forall\, w \in \Omega,$$

and in this case, $x = E_Q(g)(1+r)^{-N}$ must be the price of g.

The market consisting of shares of a company $\{S_K\}$ is said to be complete if every contingent claim admits a hedging strategy.

In a complete market, the price of every contingent claim is uniquely determined. The following results tell us when a market is complete.

Theorem: The market consisting of shares of a company evolving a $\{S_N\}$ is complete iff for all $(s_0, \ldots, s_k) \in H_L$, $0 \le k < N$.

$$\#\{s : P(S_{K+1} = s \mid S_0, \ldots, S_k = s_k) > 0\} = 1 \text{ or } 2.$$

Theorem: Assume that NA holds. The market consisting of shares of a company evolving as $\{S_k\}$ is complete iff it admits a unique EMM.

American options

Till now, we have discussed European options. These can be exercised by the buyer at the terminal time – so in case of call option, the buyer of the option contract can buy one share at price K at time N. Likewise a buyer of a put option can sell one share at price K at time N.

These options are actually traded in European markets. The options that are traded in American markets differ on a crucial point – they can be exercised at any time between buying the contract and the terminal time.

Clearly, an investor can decide to exercise his/her option at a time $k(k \le N)$ based on information available to him/her till then, namely $S_0, S_1, \ldots, S_k$. The investor is not allowed to foresee the future. Thus,

τ = first time the share price exceeds 1000 is allowed.

$$\sigma = \tau - 1, \quad \text{if } \tau \ge 1,$$
$$= 0 \quad \text{if } \tau = 0,$$

is not allowed (in practice, an investor can implement the strategy τ, but cannot implement the strategy σ: At time 5, say an investor has observed $S_0 = 890$, $S_1 = 870$, $S_2 = 930$, $S_3 = 915$, $S_4 = 965$, $S_5 = 980$. Then $\tau > 5$ and the investor knows that he must continue. However, since S_6 has not been observed, the investor cannot decide between the alternatives: $\sigma = 5$ or $\sigma > 5$.

If $(S_0, S_1, \ldots, S_k)$ denote the observations at time k, then a random variable τ is said to be a stopping time, if the event $\{\tau = k\}$ is a function of $S_0, \ldots, S_k$: having observed $S_0(w), \ldots, S_k(w)$; the investor can decide if $\{\tau(w) = k\}$ or $\{\tau(w) > k\}$.

Thus, the time at which an investor can exercise his/her option must be a stopping time.

The theory of pricing of American options has strong connections with the theory of optimal stopping times.

Upper and lower bounds for the price of an American option can be defined as in the case of European options with suitable modifications.

In case the underlying model satisfies NA and is complete, the upper and lower bounds coincide and the American option price is also uniquely determined.

It can be shown that the American call option price is the same as the corresponding European call option price:

$$E_Q\left(\frac{(S_N - K)^+}{(1+r)^N}\right),$$

where K is the strike price, N is the terminal time and Q is the EMM.

However, for the American put option, the price is strictly higher than that of the corresponding European put option: the American option price is

$$\sup_\tau E_Q\left[\frac{(K - S_\tau)^+}{(1+r)^\tau}\right],$$

where the supremum is taken over all stopping times, while the price of European put option is

$$E_Q\left(\frac{(K - S_N)^+}{(1+r)^N}\right).$$

Option pricing in continuous time models

We will now consider a more realistic scenario: where the stock prices are allowed to change continuously, and where an investor can buy or sell shares at any time. We will continue to assume that the market is frictionless.

Let $\{S_t: t \ge 0\}$ denote the shares price at time t. Let r be the instantaneous rate of return, so that an investment of Rs 1 is worth Rs e^{rt} at time t.

Let $\tilde{S}_t = S_t e^{-rt}$ denote the discounted price of the share at time t.

We will first consider simple strategies, where an investor engages in trading at finitely many time points, say at $t_0 < t_1 < \ldots < t_m$. Let a_i denote the number of shares the strategy requires the investor to hold during (t_i, t_{i+1}), $1 \le i$, (over $[0, t_1]$ for $i = 0$). The trading strategy $\{\pi_t\}$ is then represented as

$$\pi_t = a_0 1_{[0, t_1]}(t) + \sum_{i=1}^{m-1} a_i 1_{(t_i, t_{i+1})}. \quad (31)$$

While deciding on the number a_i of shares to be kept during (t_i, t_{i+1}), the investor would have observed $\{S_u: u \le t_i\}$ and thus a_i can be a function of $\{S_u: u \le t_i\}$. Thus, π_t is a function of $\{S_u: u \le t\}$ for every t. This is expressed as: $\{\pi_t\}$ is adapted to $\{S_u: u \le t\}$ in the stochastic process literature. For the simple trading strategy $\{\pi_t\}$ given by eq. (31) and initial investment x, the worth of the holdings at time t, $V_t(x, \pi)$ and its discounted value $\widetilde{V}_t(x, \pi) = e^{-rt}$ $V_t(x, \pi)$ are given by

$$\widetilde{V}_t(x,\pi) = x + \sum_{i\ t_{i+1} \le t} a_i(\widetilde{S}_{t_{i+1}} - \widetilde{S}_{t_i})$$

$$+ \sum_i a_i(\widetilde{S}_t - \widetilde{S}_{t_o}) 1_{(t_i, t_{i+1})}(t) \quad (32)$$

which can be formally written as

$$\widetilde{V}_t(x,\pi) = x + \int_0^t \pi_u \, d\widetilde{S}_u. \quad (33)$$

A simple trading strategy π is an arbitrage opportunity, if

$$P(\widetilde{V}_T(a,\pi) \ge 0) = 1,$$

and $P(\widetilde{V}_T(a,\pi) > 0) > 0$. Note that in this framework, the number of outcomes can be infinite and thus for all outcomes and for some outcomes are replaced by 'with probability one' and 'with positive probability', respectively.

Here, it can be easily shown that if there exists a probability measure Q such that P and Q are mutually absolutely continuous and such that $\widetilde{S}_t$ is a Q-Martingale, (such a Q is an EMM), then arbitrage opportunities do not exist in the class of simple strategies.

The converse however is not true. It is true after we make some technical modifications.

Thus, one assumes the existence of an EMM, Q. This, in turn, implies that S_t, $\widetilde{S}_t$ are P-semi Martingales. We can then consider limits of simple strategies as (idealized) trading strategies. The (discounted) value is still given by eq. (33) where the integral is the stochastic integral.

For further discussion of hedging strategies, upper–lower bounds, American options, etc. we would need the heavy machinery of stochastic calculus.

We hope that the discussion presented here would entice a few readers to read the relevant portion of stochastic calculus and then pursue this area of option pricing in the context of Indian markets.

1. Kallainpur, G. and Karandikar, R. L., *An Introduction to Option Pricing Theory*, Birkhauser, Boston, 2000.