Implicit Partial Differential Equations. Bernard Dacorogna and Paolo Marcellini. Birkhäuser Verlag AG, P.O. Box 133, CH-4010, Basel, Switzerland, 1999. 228 pp. Price SFr 108/DM 128.

In the theory of partial differential equations, one of the central questions is when will there be a unique solution to a given equation. This book explores the entirely opposite situation: when uniqueness fails, that too, quite dramatically. Typical theorems are proved for certain types of first-order scalar and vector equations and also for higher-order equations, giving the existence of a dense set of solutions in suitable function spaces. These equations are called 'implicit' type in this monograph. Naturally, the equations for which uniqueness results do exist, for example, first-order equations where the derivative appears linearly, are to be excluded from this class of implicit partial differential equations.

These existence results help one to prove attainment theorems for certain problems from calculus of variations involving minimization of integrals in function spaces. In such problems it is not possible to apply the usual 'direct' methods due to lack of convexity, (lack of even quasi convexity in the vector–valued case) of the integrand with respect to the gradient variable. Many such interesting applications arising from optimal design, singular values, and N-well potential are given.

The basic ideas involved in getting the existence of such dense sets of solutions, can be well illustrated in the case of the simple equation $|u'(x)| = 1$ on $[-1, 1]$, $u(-1) = u(0)$. In this context Cellina was the first to introduce this functional analytic method, to prove density of the solutions, in the space of Lipschitz continuous functions. This method was further developed by others for differential inclusions. The authors of the present book extended it to a much larger framework, adding interesting applications of such existence theorems.

In the context of the above equation, let us consider the following subspace of $W_0^{1,\infty}$, the space of Lipschitz continuous functions vanishing at -1 and 1:

$$V = \{u \in W_0^{1,\infty}(-1, 1): F(u'(x)) \leq 0 \text{ almost everywhere on } (-1, 1)\},$$

where $F(u'(x)) = |u'(x)| - 1$. This V endowed with the C^0 metric is a complete metric space because any Cauchy sequence in V has uniformly bounded gradient and therefore has a subsequence that converges weak * in $W^{1,\infty}$ to a limit which will lie in V, F being lower semicontinuous. Now introduce for every integer k,

$$V_k = \left\{ u \in V : \int_{-1}^{1} F(u'(x)) > -\frac{1}{k} \right\}.$$

By arguments similar to the above, one can show easily that V_k is open in V, while one needs deeper approximation arguments to show V_k is dense in V. From Baire Category theorem, we can then conclude that

$$\cap_k V^k = \{u \in W_0^{1,\infty}: F(u'(x)) = 0 \text{ a.e. on } (-1, 1)\},$$

is dense and hence nonempty in V.

For more general first order partial differential equations

$$\begin{cases} F(x, u(x), Du(x)) = 0 \text{ a.e. in Ù, Ù} \subset \mathbb{R}^n, \\ u = \phi \quad \text{on} \quad \partial\text{Ù}, \end{cases}$$

under mild coercivity and convexity assumptions on F, with respect to the last variable, the existence of a dense set of solutions is proved in this book, using the above method. When F is nonconvex and is independent of x and u, the p.d.e is rewritten as a differential inclusion

$$\begin{cases} Du(x) \in E \ \text{ a.e. } x \in \text{Ù}, \\ u(x) = \phi(x), \quad \text{on } x \in \partial\text{Ù}, \end{cases}$$

where $E = \{\xi \in \mathbb{R}^n \ \ F\,\xi$ – Under the compatibility condition $D\phi(x)$ lies in the interior of (convex hull (E)), this Dirichlet problem is shown to possess a dense set of solutions using a key approximation lemma to conclude the density of V_k's.

The extension of these results to the vector case where $u: \mathbb{R}^n \to \mathbb{R}^m$ is much more delicate because one needs to replace the notion of convex hull using the concepts of quasiconvexity, rank-one convexity or polyconvexity. In general for $f: \mathbb{R}^{m \times n} \to \mathbb{R}^n$,

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

When $m = 1$ or $n = 1$, all these notions are equivalent. The main problem in the vector case is that these notions are distinct and except the convex case, the other types of convex hulls are poorly understood. Another problem is the lack of good approximation theorems by piecewise affine functions. This forces one to work with piecewise C^1 functions. When the quasiconvex hulls satisfy some further structure condition, the existence of a dense set of solutions is obtained using deep approximation results.

For the second order equations of the implicit type

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0 \text{ a.e. in Ù}, \\ u = \phi, Du = D\phi \quad \text{on} \quad \partial\text{Ù}, \end{cases}$$

under a certain coercivity and convexity assumptions on F, density of solutions in the space $W^{2,\infty}$ is proved, by considering its equivalent first-order system. This coercivity condition excludes the elliptic type equations considered by Crandall–Ishii–Lions and Caffarelli–Nirenbrg–Spruck and others.

As an application, the problem of prescribed singular values of derivative matrices, $\lambda_i(Du(x))$, (the eigenvalues of $((Du)^t Du)^{1/2}$), is considered, for $u: \mathbb{R}^n \to \mathbb{R}^n$, with boundary condition $u = \phi$ on ∂Ù. Various convex hulls of E are computed and the existence of a dense set of solutions is shown under a suitable compatibility condition on ϕ. Second-order problems are also discussed. The problem of potential wells, arising in crystallographic models involving fine microstructures has also been studied and existence theorems are proved.

The book is well written and various illustrative examples help the reader to grasp the concepts well. New ideas and techniques in this fast-emerging field are well presented for the first time in a book form.

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