Three basic problems in the theory of hydrodynamic stability

H. Zhou

Department of Mechanics, Tianjin University, Tianjin 300 072, China

Three basic problems of hydrodynamic stability have been considered recently. First, the weakly nonlinear theory of hydrodynamic stability is found to yield only equilibrium solution, not transient solution, and the solution is not convergent. Second, by numerical simulation, it has been found that in supersonic boundary layers, shocklets will be induced even by small amplitude disturbances, and must therefore be considered in setting up rational theories of hydrodynamic stability for such flows. Third, a new method for inducing large-scale structures in compressible mixing layers has been suggested.

Introduction

Hydrodynamic stability is an important branch of fluid mechanics both from theoretical and application points of view. This paper presents results on three problems, which we thought were important, obtained recently by our group in Tianjin University.

First, it was found that the widely known weakly nonlinear theory of hydrodynamic stability was incapable of quantitatively explaining experimental findings, so we tried to find the reason. Eventually, we found that the original theory, though intended to deal with the evolution of disturbances, was actually good only for equilibrium states. Also, the solution, obtained in series form, does not converge absolutely.

Second, the stability of supersonic boundary layers has attracted more and more attention in recent years because of its practical importance. Extension of the nonlinear theory for incompressible flows to supersonic flows has been proposed. However, in our view, shocklets might be induced in supersonic boundary layers even when the amplitude of disturbances is still small. Numerical simulation has been carried out for a supersonic boundary layer with a small amplitude disturbance introduced into the flow, and the results show that shocklets were indeed induced. Thus, any attempt to set up a nonlinear theory for supersonic flows has to take this fact into consideration, and shock capturing schemes must be used for numerical simulations of such flows.

e-mail: hzhou@tju.edu.cn

Third, introduction of instability waves into shear flows has been proven to be effective for controlling the mixing in incompressible flows. However, for compressible flows, this method becomes less effective due to the fact that the amplification rate of the instability waves becomes smaller as the compressibility effect increases. A new method of inducing large-scale structures in compressible mixing layer has been tested by numerical simulation. 2-D test cases showed that the proposed method is indeed effective.

An analysis of the weakly nonlinear theory of hydrodynamic stability

The weakly nonlinear theory of hydrodynamic stability, first proposed by J. T. Stuart[1], is widely known. However, there are two major problems that have been overlooked by most people. First, the solution given by the theory is only good for equilibrium. Second, the equilibrium solution is always divergent. In the following, we will give a brief analysis.

We will use the formulation in real form, and will limit ourselves to the temporal problem of 2-D plane Poiseuille flow.

The Navier–Stokes equation, continuity equation and the boundary conditions read

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla \mathbf{p} + \frac{1}{R}\nabla^2 u = 0, \qquad \square$$
$$\nabla\cdot\mathbf{u} = 0, \quad \text{and} \quad \mathbf{u} = 0 \text{ at } y = \pm 1,$$

where $\mathbf{u} = \{u, v\}^{\mathrm{T}}$ is the velocity vector, u and v its components in x (stream-wise) and y (normal) directions respectively, t the time, p the pressure, ∇ the gradient operator, R the Reynolds number, all of them have been made non-dimensional.

In the weakly nonlinear theory, assume there is a small parameter ε, and the solution can be expanded as a series in ε,

$$\{\mathbf{u}, p\}^T = \{\mathbf{u}_0, p_0\}^T + \varepsilon\{\mathbf{u}_1, p_1\}^T + \varepsilon^2\{\mathbf{u}_2, p_2\}^T + \cdots, \quad (2)$$

where $\{\mathbf{u}_0, p_0\}^T$ is the basic flow and $\{\mathbf{u}_1, p_1\}^T$ the solution of the corresponding linear problem. For the temporal mode, the linear solution can be written as

$$\{\mathbf{u}_1, p_1\}^T = \{\hat{\mathbf{u}}_1(y), \hat{p}_1(y)\}^T \exp[i(\alpha x - \omega t)] + \text{c.c.}, \quad (3)$$

in which c.c. stands for complex conjugate, α is the given wave number, and $\omega = \omega_r + i\omega_i$ the complex frequency. If we normalize $\{\hat{\mathbf{u}}_1(y), \hat{p}_1(y)\}^T$ but keep the notation unchanged, then in eq. (3) there should be an amplitude parameter a. If we let the linear growth rate be absorbed into a, then we can write

$$\{\mathbf{u}_1, p_1\}^T = a(t)[\{\hat{\mathbf{u}}_1(y), \hat{p}_1(y)\}^T \exp(i\theta) + \text{c.c.}], \quad (4)$$
$$\theta = \alpha x - \omega_r t.$$

Obviously, for the linear problem, we have

$$\mathrm{d}a/\mathrm{d}t = \omega_i a, \ \mathrm{d}\theta/\mathrm{d}t = -\omega_r.$$

For the nonlinear problem, after some derivation, one would obtain the Landau eq. (5) and system of perturbation (eqs (6)) (refs 2, 3), in which A_j and B_j should be determined during the course of solution by the so-called solvability condition.

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \omega_i a + \sum_{j=3,5,\ldots} A_j a^j, \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\omega_r + \sum_{j=2,4,\ldots} B_j a^j, \quad (5)$$

$$\omega_i a\frac{\partial \mathbf{u}_n}{\partial a} + \omega_r\frac{\partial \mathbf{u}_n}{\partial \theta} + u_0\frac{\partial \mathbf{u}_n}{\partial x} + v_n\frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}y} + \nabla p_n - \frac{1}{R}\nabla^2\mathbf{u}_n$$
$$= -\sum_{\substack{j+k=n\\ j,k\geq 1}} (\mathbf{u}_j\cdot\nabla)\mathbf{u}_k - \sum_{\substack{j+k=n\\ j,k\geq 1}}\left(A_j\frac{\partial \mathbf{u}_k}{\partial a} + B_j\frac{\partial \mathbf{u}_k}{\partial \theta}\right), \quad (6)$$

$$\nabla\cdot\mathbf{u}_n = 0, \text{ and } \mathbf{u}_n = 0 \text{ at } y = \pm 1.$$

In the original theory, solutions of eq. (6) were only sought in the form of $a^n\varphi_{nm}\exp\{im\theta\}$, where m is an integer. However, it is only a particular solution. The general solution of the corresponding homogeneous equation of (6) has been neglected. Thus, the solution so obtained is not good for describing the transient process, and this fact has been overlooked by almost everybody working in this field.

At first sight, the neglect of this solution seems to be reasonable, as the corresponding homogeneous equation of (6) does not allow such solutions with integer n and m, if both of them are not equal to 1. But mathematically there is no reason why non-integer n and m should not be allowed, and if we seek the solution in the form of $a^{sn}\exp(ipm\theta)$, in which s and p are real numbers, then the corresponding homogeneous equation of (6) does have a solution, and in fact, an infinite set of eigen-solutions. Only with these solutions added is a general solution obtained that can deal with transient problems.

So the first conclusion is that using the Landau equation for describing the transient process is conceptually incorrect.

Since the solution is expressed as a series, mathematically it can offer a useful solution only if it is absolutely convergent. Otherwise the solution will depend on the order of summation, thus rendering it mathematically ambiguous. In choosing a particular partial set of the infinite series solution, we were able to derive a necessary condition for the absolute convergence of the Landau series and the solution itself. The condition reads

$$\left|\sum_{s=3,5,..}^{N} A_s a^{s-1} \Big/ \omega_\mathrm{i}\right| < 1. \quad (7)$$

If we truncate the first Landau equation at order N, and put $\mathrm{d}a/\mathrm{d}t = 0$ to solve for the equilibrium amplitude a_e, then for this a_e, we have

$$\sum_{s=3,5,..}^{N} A_s a_e^{s-1}/\omega_\mathrm{i} = 1.$$

Hence for this a_e, the Landau series cannot be absolutely convergent. We can also prove that the solution itself is not absolutely convergent.

The details of the analysis can be found in refs 2, 3, and ways for its amendment have been proposed, which seemed to be successful on comparison of the results with corresponding direct numerical simulations.

Stability problem of supersonic boundary layers[4]

Recently, the problem of transition in supersonic boundary layers has attracted more and more attention due to its technical importance. Up to now, most analyses were linear, and detailed experiments are very rare. People have tried to extend the nonlinear theory of hydrodynamic stability for incompressible flows to compressible flows. However, in our opinion, it has to be done very cautiously. Because, firstly, there is no detailed experimental observation that can help theoreticians to set up theoretical models. Secondly, for supersonic boundary layers, shocklets might be induced by the disturbance even when its amplitude is still small, which implies that discontinuities exist in the flow field, thus making any analytical method very difficult, if not impossible, to apply.

The existence of shocklets can best be verified by experiments, but this is extremely difficult. The other way is by numerical simulation. We have carried out numerical studies for a 2-D problem of a supersonic boundary layer.

The computations were made for a flat plate boundary layer with oncoming uniform flow of Mach number $M = 4.5$. First, the basic flow was calculated for a length in x-direction sufficiently long, about 75 wavelengths of the T–S wave, which was introduced later. Then we chose the flow field between the sections $x = 1030$ to 1250, measured by the boundary layer thickness at $x = 1000$, and introduced a T–S wave with a frequency $\omega = 0.37822$ at $x = 1030$, where the Reynolds number was about 1500, based on the momentum thickness at $x = 1000$ and the velocity and viscosity at infinity. The shape of the T–S wave was solved from the eigenvalue problem of the Orr-Sommerfeld equation, using the local flow profile as the basic flow. Three different amplitudes have been chosen, namely, 0.0001, 0.001, and 0.01.

For the numerical computation of the convection terms, a shock-capturing scheme must be used, because shocklets might be invoked by the disturbance. For this reason the NND scheme[5] was chosen. For the viscous terms, a second order, central difference scheme was used. The same schemes were used for the computation of the basic flow. The time derivative was computed by a 3rd order Runge–Kutta scheme.

The NND scheme expresses the first order derivative as

$$\left(\frac{\partial f^+}{\partial x}\right)_j \approx \frac{1}{\Delta x}\left\{\left[f_j^+ + \frac{1}{2}\min\mathrm{mod}\left(\Delta f^+_{j+\frac{1}{2}}, \Delta f^+_{j-\frac{1}{2}}\right)\right] - \left[f^+_{j-1} + \frac{1}{2}\min\mathrm{mod}\left(\Delta f^+_{j-\frac{1}{2}}, \Delta f^+_{j-\frac{3}{2}}\right)\right]\right\},$$

$$\left(\frac{\partial f^-}{\partial x}\right)_j \approx \frac{1}{\Delta x}\left\{\left[f^-_{j+1} - \frac{1}{2}\min\mathrm{mod}\left(\Delta f^-_{j+\frac{3}{2}}, \Delta f^-_{j+\frac{1}{2}}\right)\right] - \left[f_j^- - \frac{1}{2}\min\mathrm{mod}\left(\Delta f^-_{j+\frac{1}{2}}, \Delta f^-_{j-\frac{1}{2}}\right)\right]\right\},$$

$$\Delta f^+_{j+1/2} = f^+_{j+1} - f^+_j \qquad \Delta f^+_{j-2/3} = f^+_{j-1} - f^+_{j-2}, \text{ etc.}$$

and

$$\min\mathrm{mod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \\ b & \text{if } |b| < |a| \\ 0 & \text{if } a \times b < 0, \end{cases}$$

where the superscripts + and – imply positive and negative fluxes after flux splitting.

The NND scheme is of second order accuracy, which in general is not enough for a stability calculation. But unfortunately, to our knowledge, there is no shock capturing scheme having an accuracy higher than second order that can yet capture a shock as effectively as the NND scheme, which is also very robust.

The NND scheme, like many other shock-capturing schemes, has some sort of a so-called limiter, such as the minmod operation involved above, which chooses a concrete scheme according to the current local flow condition. Thus, for non-stationary problems, the concrete scheme at any fixed point may change from time to time, and numerical noise will be generated, contaminating the flow field. Therefore, we were not able to get useful information for the disturbances by simply extracting the quantities corresponding to the basic flow, calculated before, from the instantaneous flow quantities. Instead, the basic flow quantities must be recalculated by time averaging within a period, and actually will be different from those calculated before. For the case with the amplitude of the T–S wave being 0.0001, we were not able to get a smooth evolution curve even by such a procedure. For the other two cases, smooth evolution curves for the amplitude of the disturbances were obtained, as shown in Figure 1, in which both curves have been normalized to have the same initial value. The decay rate of the disturbance with the initial amplitude of 0.01 is appreciably larger than that with the initial amplitude of 0.001. For incompressible flow, the difference should not be so large.

Figure 2 shows the iso-Mach number lines around the critical layer for the case with amplitude 0.01. One can see shocklets around the critical layer, but they are rather weak and not so obvious. However, they may be identified by the fact that in crossing the critical layer, the periodic disturbance pressure and density become out of phase, implying an entropy jump. Yet the shocklet can only be clearly identified for the first wave cycle, and then becomes weaker and weaker for the later cycles. The reason for this is not very clear now, though it is likely to be due to numerical dissipation and the dissipation mechanism of shocklets. Also, the decay rate

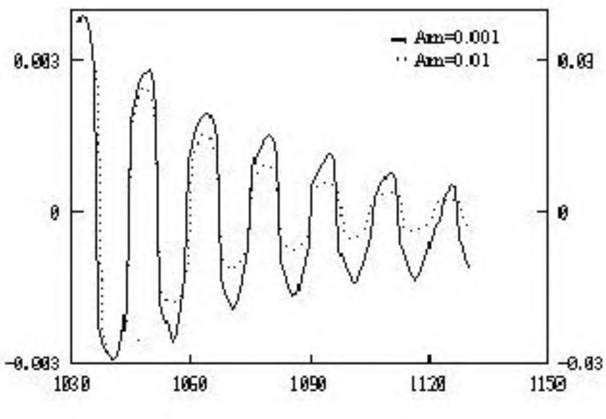


Figure 1. The evolution curve for the disturbance of density.

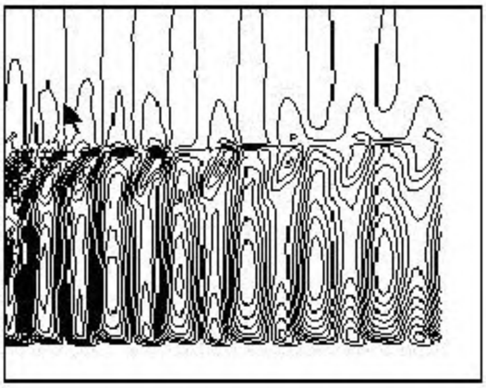


Figure 2. Iso-Mach number lines.

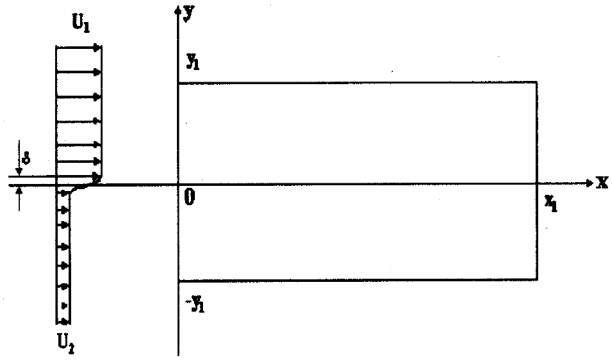


Figure 3. Domain of computation and the velocity distribution.

of the disturbance was different from that obtained by the eigenvalue problem even for the case with amplitude 0.001, because the order of accuracy of the NND scheme is not high enough.

Although our results were not accurate enough, it is clear that before any nonlinear theory of hydrodynamic stability for supersonic boundary layers can be established, more numerical work is needed. But more urgently is the need to develop a high-order-accurate numerical scheme that can capture shocklets effectively and at the same time is accurate enough to meet the requirement for stability calculation. This is not an easy task. For the same reason, one should be cautious in interpreting any results from numerical simulations for the laminar-turbulent transition as well as for the turbulence itself in supersonic flows.

A new method for enhancing the mixing of a compressible mixing layer[6]

The mixing effect of a mixing layer depends largely on its ability of generating large-scale structures due to its inherent hydrodynamic instability. However, for a compressible mixing layer, the maximum amplification rate of instability waves decreases as the compressibility effect increases, so the mixing becomes less and less effective.

Wang and Fiedler conducted an experiment[7] in which the in-flow speed at the low speed side of an incompressible mixing layer, confined in a tube, was forced to undulate. With appropriate parameters for the undulation, the mixing was greatly enhanced. Motivated by this, a numerical simulation has been carried out to see if this method was also effective for compressible flows. But due to the limitation of the computer resources, only 2-D counterpart of Wang and Fiedler's experiments has been simulated. A mixing layer with Mach number $M_e = 0.6$ at inlet centre has been systematically investigated.

The domain of computation is $[0: x_1, -y_1: y_1]$, as shown in Figure 3, in which, δ is the semi-vorticity thickness at the inlet. The in-flow speeds of the upper and lower sides are denoted by U_1 and U_2 respectively,

$$R_a = (U_1 - U_2)/(U_1 + U_2)$$

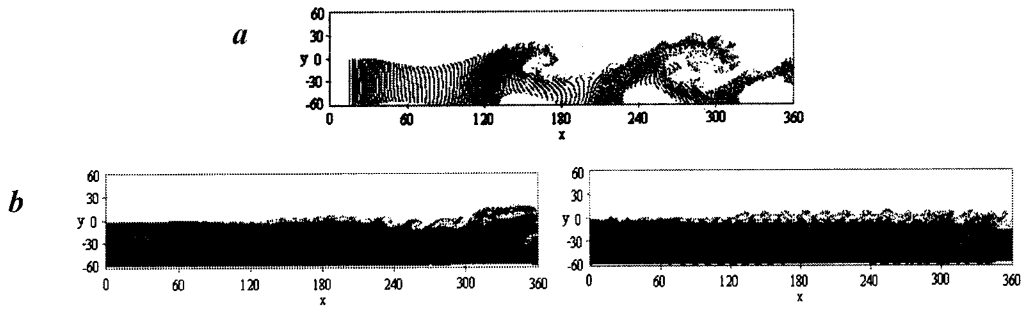
is the velocity ratio, equal to 0.24. U_e, equal to $(U_1 + U_2)/2$, ρ_e, the density at infinity, T_e, the temperature at infinity, and δ, were used as the reference quantities. The Reynolds number was defined as $R_e = \rho_e U_e \delta/\mu_e$.

In the numerical simulation, a 5th order upwind-biased compact difference scheme was used for the convection term. A 6th order centered compact scheme was used for the viscous terms. At grid points next to the boundary, the order of accuracy was reduced. The numerical boundary conditions must be able to effectively control the reflection of the waves at the boundary. The method proposed by Poinsot and Lele[8] was used for this purpose.

The accuracy and reliability of the numerical method, as well as the correctness of the boundary condition, have been checked in various ways, and the results were found to be satisfactory.

To see which method was more effective for enhancing the mixing, by forcing the inflow speed at the low speed side to undulate, or by introducing T–S waves, test cases have been computed. The basic flow parameters were the same, the amplitude of the undulation a_0 and the amplitude of the T–S wave a_T were the same as 0.1. The frequency of the undulation ω_0 was 0.05, the frequency of the T–S wave ω_T was 0.05 and 0.442, and the latter corresponded to the most unstable wave by linear stability theory. The results are shown in Figure 4 in the form of simulated smoke lines, as the scale of the generated large-scale structures could be clearly seen in this way. Obviously, for this test case, the undulation of the inflow speed is much more effective than T–S waves in enhancing the mixing.

Changing the frequency of undulation but keeping all other parameters unchanged, the simulated smoke lines are presented in Figure 5. Notice that the length in x and y directions is not in proportion.



$a_T = 0.1$, $\omega_T = 0.05$ $a_T = 0.1$, $\omega_T = 0.442$

Figure 4. Comparison of results using different ways of control. ***a***, Inflow speed undulate, $a_0 = 0.1$, $\omega_0 = 0.05$; ***b***, T–S wave introduced.

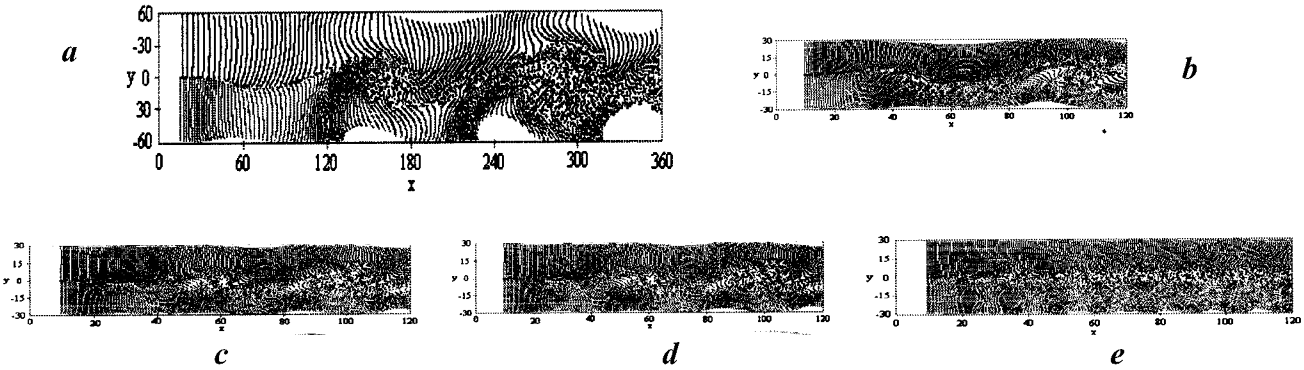


Figure 5. Smoke lines for different frequencies of undulation. ***a***, $\omega_0 = 0.03$, $x_l = 480$, $y_l = 60$; ***b***, $\omega_0 = 0.05$, $x_l = 360$, $y_l = 60$; ***c***, $\omega_0 = 0.1$, $x_l = 120$, $y_l = 30$; ***d***, $\omega_0 = 0.15$, $x_l = 120$, $y_l = 30$; ***e***, $\omega_0 = 0.3$, $x_l = 120$, $y_l = 30$.

From Figure 5 we can infer that the lower the frequency, the larger the scale of generated structures would be. But as the frequency was lowered from 0.05 to 0.03, the size of the structure seemed to experience no change. Physically, when the frequency approaches zero, there could be no structure generated in any finite domain. Therefore, for any practical equipment, there must be an optimal frequency. For our case, 0.05 seems to be the optimal value.

The amplitude of the undulation has also been changed, with other parameters kept the same. The size of the large-scale structures increased as the amplitude increased. But it was not sensitive to change of amplitude in the amplitude range 0.05 to 0.08.

The size of the generated structure was not sensitive to change of Reynolds number.

From what we have obtained by numerical simulations, we can infer that the mixing effect of a compressible mixing layer can indeed be appreciably enhanced, if the inflow speed at the lower speed side of the layer is made to undulate. Among all parameters, the frequency and amplitude are more important than others.

The supersonic mixing layer has also been tested, qualitatively, and the conclusion remains the same.

1. Stuart, J. T., *J. Fluid Mech.*, 1960, **2**, 353–370.
2. Zhou, H. and You, X., *Acta Mechanica Sinica (English Series)*, 1993, **9**, 1–12.
3. Zhou, H. and Fujimura, K., *Science in China*, 1998, **A41**, 84–92.
4. Yuan, X. and Zhou, H., Proceedings of the IUTAM Symposium for Laminar–Turbulent Transition, 1999.
5. Zhang, H. X., *J. Aerodynamics (in Chinese)*, 1988, **6**, 143–165.
6. Zhang, Z., Luo, J. and Zhou, H., *Science in China*, 2000, **43**.
7. Wang, G. R. and Fiedler, H., Second International Conference on Flow Interaction cum Exhibition/Lectures on Interaction of Science and Art, Berlin, 1997, pp. 12–14.
8. Poinsot, T. J. and Lele, S. K., *J. Computat. Phys.*, 1992, 104–129.

ACKNOWLEDGEMENT. This work was a part of the project supported by the National Natural Science Foundation of China (grant no. 19732005) and the National Climbing Project.