Sandpile models of self-organized criticality

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Self-organized criticality is the emergence of long-ranged spatio-temporal correlations in non-equilibrium steady states of slowly driven systems without fine tuning of any control parameter. Sandpiles were proposed as prototypical examples of self-organized criticality. However, only some of the laboratory experiments looking for the evidence of criticality in sandpiles have reported a positive outcome. On the other hand, a large number of theoretical models have been constructed that do show the existence of such a critical state. We discuss here some of the theoretical models as well as some experiments.

THE concept of self-organized criticality (SOC) was introduced by Bak, Tang and Wiesenfeld (BTW) in 1987 (ref. 1). It says that there is a certain class of systems in nature whose members become critical under their own dynamical evolutions. An external agency drives the system by injecting some mass (in other examples, it could be the slope, energy or even local voids) into it. This starts a transport process within the system: Whenever the mass at some local region becomes too large, it is distributed to the neighbourhood by using some local relaxation rules. Globally, mass is transported by many such successive local relaxation events. In the language of sandpiles, these together constitute a burst of activity called an avalanche. If we start with an initial uncritical state, initially most of the avalanches are small, but the range of sizes of avalanches grows with time. After a long time, the system arrives at a critical state, in which the avalanches extend over all length and time scales. Customarily, critical states have measure zero in the phase space. However, with self-organizing dynamics, the system finds these states in polynomial times, irrespective of the initial state[2–4].

BTW used the example of a sandpile to illustrate their ideas about SOC. If a sandpile is formed on a horizontal circular base with any arbitrary initial distribution of sand grains, a sandpile of fixed conical shape (steady state) is formed by slowly adding sand grains one after another (external drive). The surface of the sandpile in the steady state, on the average, makes a constant angle known as the angle of repose, with the horizontal plane. Addition of each sand grain results in some activity on the surface of the pile: an avalanche of sand mass follows, which propagates on the surface of the sandpile. Avalanches are of many different sizes and BTW argued that they would have a power law distribution in the steady state.

There are also some other naturally occurring phenomena which are considered to be examples of SOC. Slow creeping of tectonic plates against each other results in intermittent burst of stress release during earthquakes. The energy released is known to follow power law distributions as described by the well-known Gutenberg–Richter Law[5]. The phenomenon of earthquakes is being studied using SOC models[6]. River networks have been found to have fractal properties. Water flow causes erosion in river beds, which in turn changes the flow distribution in the network. It has been argued that the evolution of river pattern is a self-organized dynamical process[7]. Propagation of forest fires[8] and biological evolution processes[9] have also been suggested to be examples of SOC.

Laboratory experiments on sandpiles, however, have not always found evidence of criticality in sandpiles. In the first experiment, the granular material was kept in a semicircular drum which was slowly rotated about the horizontal axis, thus slowly tilting the free surface of the pile. Grains fell vertically downward and were allowed to pass through the plates of a capacitor. Power spectrum analysis of the time series for the fluctuating capacitance however showed a broad peak, contrary to the expectation of a power law decay, from the SOC theory[10].

In a second experiment, sand was slowly dropped on to a horizontal circular disc, to form a conical pile in the steady state. On further addition of sand, avalanches were created on the surface of the pile, and the outflow statistics was recorded. The size of the avalanche was measured by the amount of sand mass that dropped out of the system. It was observed that the avalanche size distribution obeys a scaling behaviour for small piles. For large piles, however, scaling did not work very well. It was suggested that SOC behaviour is seen only for small sizes, and very large systems would not show SOC[11].

Another experiment used a pile of rice between two vertical glass plates separated by a small gap. Rice grains were slowly dropped on to the pile. Due to the anisotropy of grains, various packing configurations were observed. In the steady state, avalanches of moving rice grains refreshed the surface repeatedly. SOC behaviour was observed for grains of large aspect ratio, but not for the less elongated grains[12].

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Theoretically, however, a large number of models have been proposed and studied. Most of these models study the system using cellular automata where discrete or continuous variables are used for the heights of sand columns. Among them, the Abelian Sandpile Model (ASM) is the most popular[1,13]. Other models of SOC have been studied but will not be discussed here. These include the Zhang model which has modified rules for sandpile evolution[14], a model for Abelian distributed processors and other stochastic rule models[15], the Eulerian Walkers model[16] and the Takayasu aggregation model[17].

In the ASM, we associate a non-negative integer variable h representing the height of the 'sand column' with every lattice site on a d-dimensional lattice (in general on any connected graph). One often starts with an arbitrary initial distribution of heights. Grains are added one at a time at randomly selected sites $O: h_O \rightarrow h_O + 1$. The sand column at any arbitrary site i becomes unstable when h_i exceeds a previously selected threshold value h_c for the stability. Without loss of generality, one usually chooses $h_c = 2d - 1$. An unstable sand column always topples. In a toppling, the height is reduced as: $h_i \rightarrow h_i - 2d$ and all the $2d$ neighbouring sites $\{j\}$ gain a unit sand grain each: $h_j \rightarrow h_j + 1$. This toppling may make some of the neighbouring sites unstable. Consequently, these sites will topple again, possibly making further neighbours unstable. In this way a cascade of topplings propagates, which finally terminates when all sites in the system become stable (Figure 1). One waits until this avalanche stops before adding the next grain. This is equivalent to assuming that the rate of adding sand is much slower than the natural rate of relaxation of the system. The wide separation of the 'time scale of drive' and 'time scale of relaxation' is common in many models of SOC. For instance, in earthquakes, the drive is the slow tectonic movement of continental plates, which occurs over a time scale of centuries, while the actual stress relaxation occurs in quakes, whose duration is only a few seconds. This separation of time scales is usually considered to be a defining characteristic of SOC. However, Dhar has argued that the wide separation of time scales should not be considered as a necessary condition for SOC in general[4]. Finally, the system must have an outlet, through which the grains go out of the system, which is absolutely necessary to attain a steady state. Most popularly, the outlet is chosen as the $(d-1)$-dimensional surface of a d-dimensional hypercubic system.

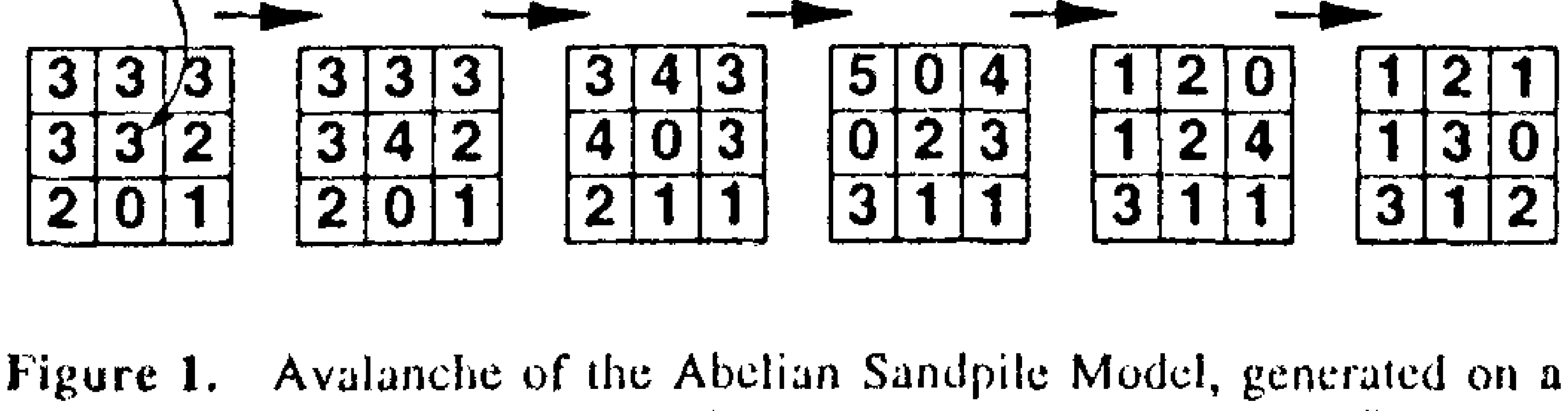

Figure 1. Avalanche of the Abelian Sandpile Model, generated on a 3×3 square lattice. A sand grain is dropped on a stable configuration at the central site. The avalanche created has size $s = 6$, area $a = 6$, life-time $t = 4$ and the radius $r = \sqrt{2}$.

The beauty of the ASM is that the final stable height configuration of the system is independent of the sequence in which sand grains are added to the system to reach this stable configuration[13]. On a stable configuration C, if two grains are added, first at i and then at j, the resulting stable configuration C' is exactly the same in case the grains were added first at j and then at i. In other sandpile models, where the stability of a sand column depends on the local slope or the local Laplacian, the dynamics is not Abelian, since toppling of one unstable site may convert another unstable site to a stable site (Figure 2). Many such rules have been studied in the literature[18,19].

An avalanche is a cascade of topplings of a number of sites created on the addition of a sand grain. The strength of an avalanche in general, is a measure of the effect of the external perturbation created due to the addition of the sand grain. Quantitatively, the strength of an avalanche is estimated in four different ways: (i) size (s): the total number topplings in the avalanche, (ii) area (a): the number of distinct sites which toppled, (iii) life-time (t): the duration of the avalanche, and (iv) radius (r): the maximum distance of a toppled site from the origin. These four different quantities are not independent and are related to each other by scaling laws. Between any two measures $x, y \in \{s, a, t, r\}$ one can define a mutual dependence as: $\langle y \rangle \sim x^{\gamma_{xy}}$. These exponents are related to one another, e.g. $\gamma_{ts} = \gamma_{tr}\,\gamma_{rs}$. For the ASM, it can be proved that the avalanche clusters cannot have any holes. It has been shown that $\gamma_{rs} = 2$ in two-dimensions. It has also been proved that $\gamma_{rt} = 5/4$ (ref. 21). A better way to estimate the γ_{tx} exponents is to average over the intermediate values of the size, area and radius at every intermediate time step during the growth of the avalanche.

Quite generally, the finite size scaling form for the probability distribution function for any measure $x \in \{s, a, t, r\}$ is taken to be:

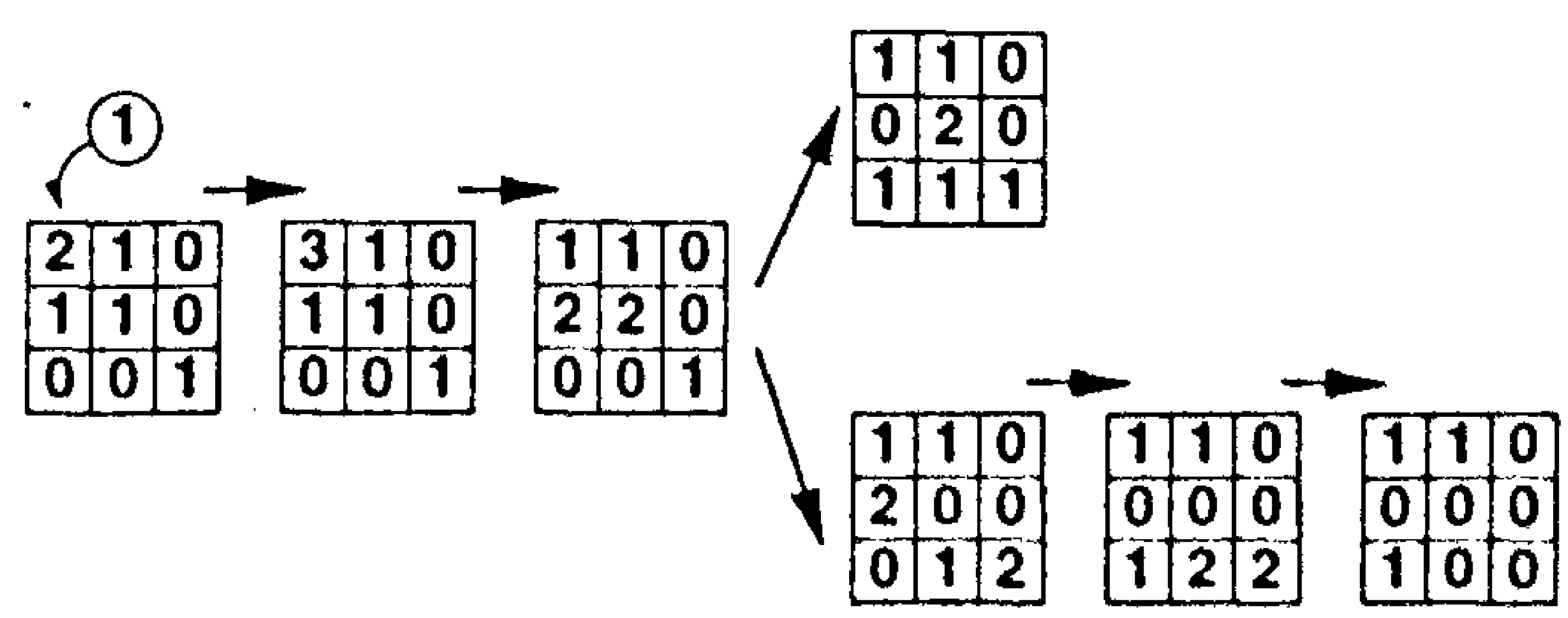

Figure 2. Example to show that a directed slope model is non-Abelian. Two slopes are measured from any site (i, j) as $h(i, j) - h(i, j+1)$ and $h(i, j) - h(i+1, j+1)$. If either of them is greater than 1, two grains are transferred from (i, j) and are given one each to $(i, j+1)$ and $(i+1, j+1)$. On dropping a grain on the initial stable configuration, we see that finally two different height configuration result due to two different sequences of topplings[20].

$$P(x) \sim x^{-\tau_x} f_x(x/L^{\sigma_x}).$$

The exponent σ_x determines the variation of the cut-off of the quantity x with the system size L. Alternatively, sometimes it is helpful to consider the cumulative probability distribution $F(x) = \int_x^{L^{\sigma_x}} P(x)\,dx$ which varies as $x^{1-\tau_x}$. However, in the case of $\tau_x = 1$, the variation should be in the form $F(x) = C - \log(x)$. Between any two measures, scaling relations like $\gamma_{xy} = (\tau_x - 1)/(\tau_y - 1)$ exist. Recently, the scaling assumptions for the avalanche sizes have been questioned. It has been argued that there actually exists a multifractal distribution instead[22].

Numerical estimation for the exponents has yielded scattered values. For example estimates of the exponent τ_s range from 1.20 (ref. 18) to 1.27 (ref. 23) and 1.29 (ref. 24).

We will now look into the structure of avalanches in more detail. A site i can topple more than once in the same avalanche. The set of its neighbouring sites $\{j\}$, can be divided into two subsets. Except at the origin O, where a grain is added from the outside, for a toppling, the site i must receive some grains from some of the neighbouring sites $\{j_1\}$ to exceed the threshold h_c. These sites must have toppled before the site i. When the site i topples, it loses $2d$ grains to the neighbours, by giving back the grains it has received from $\{j_1\}$, and also donating grains to the other neighbours $\{j_2\}$. Some of these neighbours may topple later, which returns grains to the site i and its height h_i is raised. The following possibilities may arise: (i) some sites of $\{j_2\}$ may not topple at all; then the site i will never re-topple and is a singly toppled site on the surface of the avalanche. (ii) all sites in $\{j_2\}$ topple, but no site in $\{j_1\}$ topples again; then i will be a singly toppled site, surrounded by singly toppled sites. (iii) all sites in $\{j_2\}$ topple, and some sites of $\{j_1\}$ re-topple; then i will remain a singly toppled site, adjacent to the doubly toppled sites. (iv) all sites in $\{j_2\}$ topple, and all sites of $\{j_1\}$ re-topple; then the site i must be a doubly toppled site. This implies that the set of at least doubly toppled sites must be surrounded by the set of singly toppled sites. Arguing in a similar way will reveal that sites which toppled at least n times, must be a subset and also are surrounded by the set of sites which toppled at least $(n-1)$ times. Finally, there will be a central region in the avalanche, where all sites have toppled a maximum of m times. The origin of the avalanche O where the sand grain was dropped, must be a site in this maximum toppled zone. Also, the origin must be at the boundary of this mth zone, since otherwise it should have toppled $(m+1)$ times[25].

Using this idea, we see that the boundary sites on any arbitrary system can topple at most once in any arbitrary number of avalanches. Similar restrictions are true for inner sites also. A $(2n+1) \times (2n+1)$ square lattice can be divided into $(n+1)$ subsets which are concentric squares. Sites on the mth such square from the boundary can topple at most m times, whereas the central site cannot topple more than n times in any avalanche.

Avalanches can also be decomposed in a different way, using *Waves of Toppling*. Suppose, on a stable configuration C a sand grain is added at the site O. The site is toppled once, but is not allowed to topple for the second time, till all other sites become stable. This is called the first wave. It may happen that after the first wave, the site O is stable; in that case the avalanche has terminated. If the O is still unstable, it is toppled for the second time and all other sites are allowed to become stable again; this is called the second wave, and so on. It was shown, that in a sample where all waves occur with equal weights, the probability of occurrence of a wave of area a is $D(a) \sim 1/a$ (ref. 26).

It is known that the stable height configurations in ASM are of two types: *Recurrent* configurations appear only in the steady state with uniform probabilities, whereas *Transient* configurations occur in the steady state with zero probability. Since long-range correlations appear only in the steady states, it implies that the recurrent configurations are correlated. This correlation is manifested by the fact that certain clusters of connected sites with some specific distributions of heights never appear in any recurrent configuration. Such clusters are called the forbidden sub-configurations. It is easy to show that two zero heights at the neighbouring sites: (0–0) or, an unit height with two zero heights at its two sides: (0–1–0) never occur in the steady state. There are also many more forbidden sub-configurations of bigger sizes.

An $L \times L$ lattice is a graph, which has all the sites and all the nearest neighbour edges (bonds). A *Spanning tree* is a connected sub-graph having all sites but no loops. Therefore, between any pair of sites there exists an unique path through a sequence of bonds. There can be many possible Spanning trees on a lattice. These trees have interesting statistics in a sample where they are equally likely. Suppose we randomly select such a tree and then randomly select one of the unoccupied bonds and occupy it, it forms a loop of length ℓ. It has been shown that these loops have the length distribution $D(\ell) \sim \ell^{-8/5}$. Similarly, if a bond of a Spanning tree is randomly selected and deleted, then it divides into two fragments. The sizes of the two fragments generated follow a probability distribution $D(a) \sim a^{-11/8}$ (ref. 27). It was also shown that every recurrent configuration of the ASM on an arbitrary lattice has a one-to-one correspondence to a random Spanning tree graph on the same lattice. Therefore, there are exactly the same number of distinct Spanning trees as the number of recurrent ASM configurations on any arbitrary lattice[21]. Given a stable height configuration, there exists an unique prescription to obtain the equivalent Spanning tree. This is called the *Burning* method[21]. A fire front, initially at every site outside the boundary, gradually penetrates (burns) into the system using a deterministic rule. The paths of the fire front constitute the Spanning tree. A fully burnt system is recurrent, otherwise it is transient (Figure 3).

Suppose, addition of a grain at the site O of a stable recurrent configuration C, leads to another stable configuration C'. Is it possible to get back the configuration C knowing C' and the position of O? This is done by *Inverse toppling*[28]. Since C' is recurrent, a corresponding Spanning tree ST(C') exists. Now, one grain at O is taken out from C' and the configuration $C'' = C' - \delta_{oj}$ is obtained. This means on ST(C'), one bond is deleted at O and it is divided into two fragments. Therefore one cannot burn the configuration C''completely since the resulting

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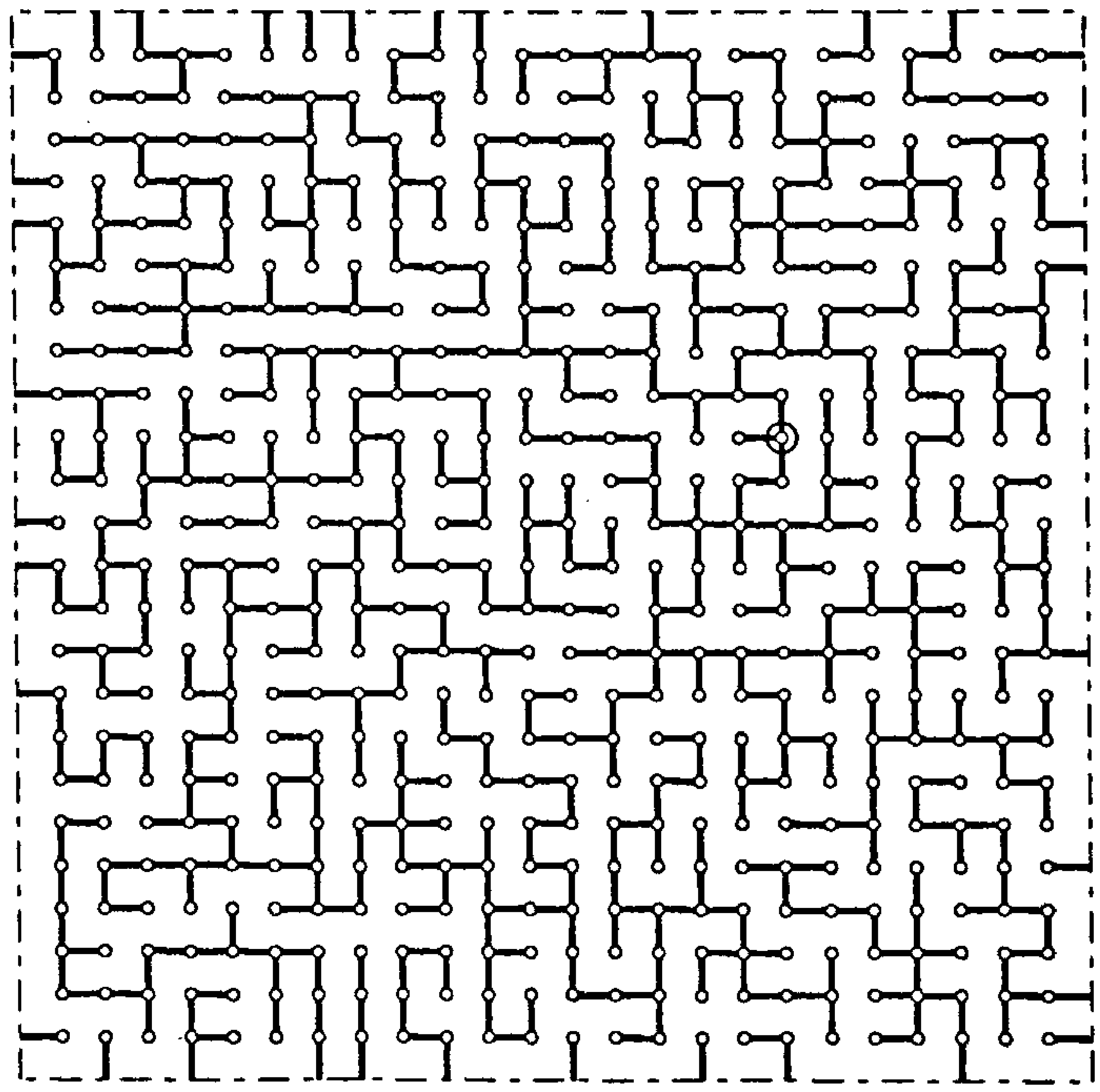
a



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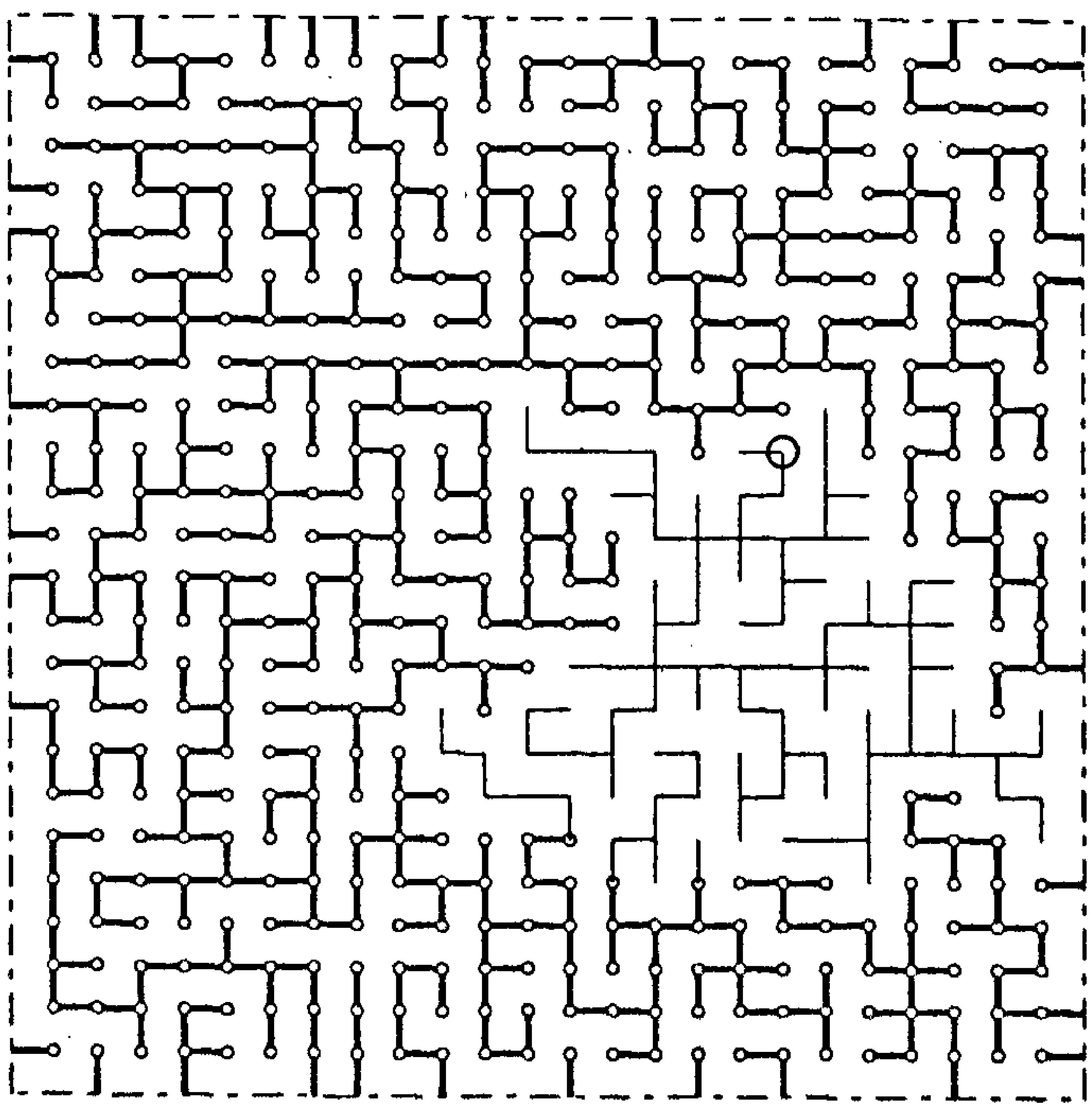

c

Figure 3. *a*, An example of the height distribution in a recurrent configuration C' on a 24 × 24 square lattice. This configuration is obtained by dropping a grain *a* some previous configuration C at the encircled site; *b*, The spanning tree representation of the configuration C'; *c*, A new configuration C'' is obtained by taking out one grain at the encircled site from the configuration C'. A spanning tree cannot be obtained for C''. The bonds of the spanning tree corresonding to the forbidden sub-configuration in C'' are shown by the thin lines.

tree has a hole consisting of at least the sites of the smaller fragment. This implies that C'' has a forbidden sub-configuration (F_1) of equal size and C'' is not recurrent. On (F_1), one runs the inverse toppling process: 4 grains are added to each site i, and one grain each is taken out from all its neighbours $\{j\}$. The cluster of f_1 sites in F_1 is called the first inverse avalanche. The lattice is burnt again. If it still has a forbidden sub-configuration (F_2), another inverse toppling process is executed, and is called the second inverse avalanche. The size of the avalanche is: $s = f_1 + f_2 + f_3 + \ldots$, and f_1 is related to the maximum toppled zone of the avalanche. From the statistics of random spanning trees[27] it is clear that f_1 should have the same statistics of the two fragments of the tree generated on deleting one bond. Therefore, the maximum toppled zone also has a power law distribution of the size, $D(a) \sim a^{-11/8}$.

Sandpile models with stochastic evolution rules have also been studied. The simplest of these is a *two-state* sandpile model. A stable configuration of this system consists of sites, either vacant or occupied by at most one grain. If there are two or more grains at a site at the same time we say there is a *collision*. In this case, all grains at that site are moved. Each grain chooses a randomly selected site from the neighbours and is moved to that site. The avalanche size is the total number of collisions in an avalanche. From the numerical simulations, the distribution of avalanche sizes is found to follow a power law, characterized by an exponent $\tau_s \approx 1.27$ (ref. 29). This two-state model has a nontrivial dynamics even in one-dimension[30]. Recently, it has been shown that instead of moving all grains, if only two grains are moved randomly leaving others at the site, the dynamics is Abelian[31].

Some other stochastic models also have nontrivial critical behaviour in one dimension. To model the dynamics of rice piles, Christensen *et al.*[32] studied the following slope model. On a one-dimensional lattice of length L, non-negative integer variable h_i represents the height of the sand column at the site i. The local slope $z_i = h_i - h_{i+1}$ is defined, maintaining zero height on the right boundary. Grains are added only at the left boundary $i = 1$. Addition of one grain $h_i \rightarrow h_{i+1}$ implies an increase in the slope $z_i \rightarrow z_i + 1$. If at any site, the local slope exceeeds a pre-assigned threshold value z_i^c, one grain is transferred from the column at i to the column at $(i + 1)$. This implies a change in the local slope as: $z_i \rightarrow z_i - 2$ and $z_{i\pm 1} \rightarrow z_{i\pm 1} + 1$. The thresholds of the instability z_i^c are dynamical variables and are randomly chosen between 1 and 2 in each toppling. Numerically, the avalanche sizes are found to follow a power law distribution with an exponent $\tau_s \approx 1.55$ and the cutoff exponent was found to be $\sigma_s \approx 2.25$. This model is referred as the Oslo model.

Addition of one grain at a time, and allowing the system to relax to its stable state, implies a zero rate of driving of the system. What happens when the driving rate is finite? Corral and Paczuski studied the Oslo model in the situation of nonzero flow rate. Grains were added at a rate r, i.e. at every $(1/r)$ time updates, one grain is dropped at the left boundary $i = 1$. They observed a dynamical transition separating intermittent and continuous flows[33].

Many different versions of the sandpile model have been studied. However the precise classification of various models in different universality classes in terms of their critical exponents is not yet available and still attracts much attention[18,19]. Exact values of the critical exponents of the most widely studied ASM are still not known in two-dimensions. Some effort has also been made towards the analytical calculation of avalanche size exponents[34–36]. Numerical studies for these exponents are found to give scattered values. On the other hand, the two-state sandpile model is believed to be better behaved and there is good agreement of numerical values of its exponents by different investigators. However, whether the ASM and the two-state model belong to the same universality class or not is still an unsettled question[37].

If a real sandpile is to be modelled in terms of any of these sandpile models or their modifications, it must be a slope model, rather than a height model. However, not much work has been done to study the slope models of sandpiles[18,19]. Another old question is whether the conservation of the grain number in the toppling rules is a necessary condition to obtain a critical state. It has been shown already that too much non-conservation leads to avalanches of characteristic sizes[36]. However, if grains are taken out of the system slowly, the system is found to be critical in some situations. A non-conservative version of the ASM with directional bias shows a mean field type critical behaviour[39]. Therefore, the detailed role of the conservation of the grain numbers during the topplings is still an open question.

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