

Persistence in nonequilibrium systems

Satya N. Majumdar

Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

This is a brief review of recent theoretical efforts to understand persistence in nonequilibrium systems.

THE problem of *persistence* in spatially extended nonequilibrium systems has recently generated a lot of interest both theoretically¹⁻⁷ and experimentally⁸⁻¹⁰. Persistence is simply the probability that the local value of the fluctuating nonequilibrium field does not change sign up to time t . It has been studied in various systems, including several models undergoing phase separation^{1-4,11-15}, the simple diffusion equation with random initial conditions^{5,6}, several reaction diffusion systems in both pure¹⁶ and disordered¹⁷ environments, fluctuating interfaces¹⁸⁻²⁰, Lotka–Volterra models of population dynamics²¹, and granular media²².

The precise definition of persistence is as follows. Let $\phi(x, t)$ be a nonequilibrium field fluctuating in space and time according to some dynamics. For example, it could represent the coarsening spin field in the Ising model after being quenched to low temperature from an initial high temperature. It could also be simply a diffusing field starting from random initial configuration or the height of a fluctuating interface. Persistence is simply the probability $P_0(t)$ that at a fixed point in space, the quantity $\text{sgn}[\phi(x, t) - \langle \phi(x, t) \rangle]$ does not change up to time t . In all the examples mentioned above this probability decays as a power law $P_0(t) \sim t^{-\theta}$ at late times, where the persistence exponent θ is usually nontrivial.

In this article, we review some recent theoretical efforts in calculating this nontrivial exponent in various models and also mention some recent experiments that measured this exponent. The plan of the paper is as follows. We first discuss the persistence in very simple single variable systems. This makes the ground for later study of persistence in more complex many-body systems. Next, we consider many-body systems such as the Ising model and discuss where the complexity is coming from. We follow it up with the calculation of this exponent for a simpler many-body system namely diffusion equation and see that even in this simple case, the exponent θ is nontrivial. Next, we show that all these examples can be viewed within the general framework of the ‘zero crossing’ problem of a Gaussian stationary process (GSP).

We review the new results obtained for this general Gaussian problem in various special cases. Finally, we mention the emerging new directions towards different generalizations of persistence.

We start with a very simple system namely the one-dimensional Brownian walker. Let $\phi(t)$ represent the position of a 1-D Brownian walker at time t . This is a single-body system in the sense that the field ϕ has no x dependence but only t dependence. The position of the walker evolves as,

$$\frac{d\phi}{dt} = \eta(t), \quad (1)$$

where $\eta(t)$ is a white noise with zero mean and delta correlated, $\langle \phi(t)\phi(t') \rangle = \delta(t - t')$. Then persistence $P_0(t)$ is simply the probability that $\phi(t)$ does not change sign up to time t , i.e. the walker does not cross the origin up to time t . This problem can be very easily solved exactly by writing down the corresponding Fokker–Planck equation with an absorbing boundary condition at the origin²³. The persistence decays as $P_0(t) \sim t^{-1/2}$ and hence $\theta = 1/2$. The important point to note here is that the exact calculation is possible here due to the Markovian nature of the process in eq. (1). Note that ϕ evolves according to a first order equation in time, i.e. to know $\phi(t)$, we just need the value of $\phi(t - \Delta t)$ but not on the previous history. This is precisely the definition of a Markov process.

In order to make contact with the general framework to be developed in this article, we now solve the same process by a different method. We note from eq. (1) that $\eta(t)$ is a Gaussian noise and eq. (1) is linear in ϕ . Hence, ϕ is also a Gaussian process with zero mean and a two-time correlator, $\langle \phi(t)\phi(t') \rangle = \min(t, t')$ obtained by integrating eq. (1). We recall that a Gaussian process can be completely characterized by just the two-time correlator. Any higher order correlator can be simply calculated by using Wick’s theorem. Since $\min(t, t')$ depends on both time t and t' and not just on their difference $|t - t'|$, clearly ϕ is a Gaussian non-stationary process. From the technical point of view, stationary processes are often preferable to non-stationary processes. Fortunately there turns out to be a simple transformation by which one can convert this non-stationary process into a stationary one. It turns out that this transformation is more general and will work even for more complicated examples to follow. Therefore we illustrate it in detail for the Brownian walker problem in the following paragraph.

The transformation works as follows. Consider first the normalized process, $X(t) = \phi(t)/\sqrt{\langle \phi^2(t) \rangle}$. Then, $X(t)$ is also a Gaussian process with zero mean and its two-time correlator is given by, $\langle X(t)X(t') \rangle = \min(t, t')/\sqrt{tt'}$. Now we define a new 'time' variable, $T = \log(t)$. Then, in this new time variable T , the two-time correlator becomes, $\langle X(T)X(T') \rangle = \exp(-|T - T'|/2)$ and hence is stationary in T . Thus, the persistence problem reduces to calculating the probability $P_0(T)$ of no zero crossing of $X(T)$, a GSP characterized by its two-time correlator, $\langle X(T)X(T') \rangle = \exp(-|T - T'|/2)$.

One could, of course, ask the same question for an arbitrary GSP with a given correlator $\langle X(T)X(T') \rangle = f(|T - T'|)$ [in case of Brownian motion, $f(T) = \exp(-T/2)$]. This general zero crossing problem of a GSP has been studied by mathematicians for a long time²⁴. Few results are known exactly. For example, it is known that if $f(T) < 1/T$ for large T , then $P_0(T) \sim \exp(-\mu T)$ for large T . Exact result is known only for Markov GSP which are characterized by purely exponential correlator, $f(T) = \exp(-\lambda T)$. In that case, $P_0(T) = (2/\pi) \sin^{-1}[\exp(-\lambda T)]$ (ref. 24). Our example of Brownian motion corresponds to the case when $\lambda = 1/2$ and therefore the persistence $P_0(T) \sim \exp(-T/2)$ for large T . Reverting to the original time using $T = \log(t)$, we recover the result, $P_0(t) \sim t^{-1/2}$. Thus the inverse of the decay rate in T becomes the power law exponent in t by virtue of this 'log-time' transformation. Note that when the correlator $f(T)$ is different from pure exponential, the process is non-Markovian and in that case no general answer is known.

Having described the simplest one-body Markov process, we now consider another one-body process which however is non-Markovian. Let $\phi(t)$ (still independent of x) now represent the position of a particle undergoing random acceleration,

$$\frac{d^2\phi}{dt^2} = \eta(t), \quad (2)$$

where $\eta(t)$ is a white noise as before. What is the probability $P_0(t)$ that the particle does not cross zero up to time t ? This problem was first proposed in the review article by Wang and Uhlenbeck²⁵ way back in 1945 and it got solved only very recently in 1992, first by Sinai²⁶, followed by Burkhardt²⁷ by a different method. The answer is, $P_0(t) \sim t^{-1/4}$ for large t and the persistence exponent is $\theta = 1/4$. Thus even for this apparently simple looking problem, the calculation of θ is nontrivial. This nontriviality can be traced back to the fact that this process is non-Markovian. Note that eq. (2) is a second order equation and to know $\phi(t + \Delta t)$, we need to know its values at two previous points $\phi(t)$ and $\phi(t - \Delta t)$. Thus, it depends on two previous steps as opposed to just the previous step as in eq. (1). Hence it is a non-Markovian process.

We notice that eq. (2) is still linear and hence $\phi(t)$ is still a Gaussian process with a non-stationary correlator. However, using the same $T = \log(t)$ transformation as defined in the previous paragraph, we can convert this to the zero crossing problem in time T of a GSP with correlator, $f(T) = \frac{3}{2} \exp(-T/2) - \frac{1}{2} \exp(-3T/2)$. Note that this is different from pure exponential and hence is non-Markovian. We also notice another important point: It is not correct to just consider the asymptotic form of $f(T) \sim \frac{3}{2} \exp(-T/2)$ and conclude that the exponent is therefore 1/2. The fact that the exponent is exactly 1/4, reflects that the 'no zero crossing' probability $P_0(T)$ depends very crucially on the full functional form of $f(T)$ and not just on its asymptotic form. This example thus illustrates the history dependence of the exponent θ which makes its calculation nontrivial.

Having discussed the single particle system, let us now turn to many body systems where the field $\phi(x, t)$ now has x dependence also. The first example that was studied is when $\phi(x, t)$ represents the spin field of one-dimensional Ising model undergoing zero temperature coarsening dynamics, starting from a random high temperature configuration. Let us consider for simplicity a discrete lattice where $\phi(i, t) = \pm 1$ representing Ising spins. One starts from a random initial configuration of these spins. The zero temperature dynamics proceeds as follows: at every step, a spin is chosen at random and its value is updated to that of one of its neighbours chosen at random and then time is incremented by Δt and one keeps repeating this process. Then persistence is simply the probability that a given spin (say at site i) does not flip up to time t . Even in one dimension, the calculation of $P_0(t)$ is quite nontrivial. Derrida *et al.*² solved this problem exactly and found $P_0(t) \sim t^{-\theta}$ for large t with $\theta = 3/8$. They also generalized this to q -state Potts model in 1-D and found an exact formula, $\theta(q) = -\frac{1}{q} + \frac{2}{\pi^2} [\cos^{-1}\{(2-q)\sqrt{2-q}\}]^2$ for all q .

This calculation however cannot be easily extended to $d = 2$ which is more relevant from an experimental point of view. Early numerical results indicated that the exponent $\theta \sim 0.22$ (ref. 3) for $d = 2$ Ising model evolving with zero temperature spin flip dynamics. It was therefore important to have a theory in $d = 2$ which, if not exact, at least could give approximate results. We will discuss later about our efforts towards such an approximate theory of Ising model in higher dimensions. But before that let us try to understand the main difficulties that one encounters in general in many-body systems.

In a many-body system, if one sits at a particular point x in space and monitors the local field $\phi(x, t)$ there as a function of t , how would this 'effective' stochastic process (as a function of time only) look like? If one knows enough properties of this single site process as a function of time, then the next step is to ask what is the probability that this stochastic process viewed from x as a function of t , does not change sign up to time t . So, the general

strategy involves two steps: first, one has to solve the underlying many-body dynamics to find out what the 'effective' single site process looks like and second, given this single site process, what is its no zero crossing probability.

Before discussing the higher dimensional Ising model where both of these steps are quite hard, let us discuss a simple example (which however is quite abundant in nature), namely the diffusion equation. This is a many-body system but at least the first step of the two-step strategy can be carried out exactly and quite simply. The second step cannot be carried out exactly even for this simple example, but one can obtain very good approximate results.

Let $\phi(x, t)$ (which depends on both x and t) denote field that is evolving via the simple diffusion equation,

$$\frac{\partial \phi}{\partial t} = \nabla^2 \phi. \quad (3)$$

This equation is deterministic and the only randomness is in the initial condition $\phi(x, 0)$ which can be chosen as a Gaussian random variable with zero mean. For example, $\phi(x, t)$ could simply represent the density fluctuation, $\phi(x, t) = \rho(x, t) - \langle \rho \rangle$ of a diffusing gas. The persistence, as usual, is simply the probability that $\phi(x, t)$ at some x does not change sign up to time t . This classical diffusion equation is so simple that it came as a surprise to find that even in this case, the persistence $P_0(t) \sim t^{-\theta}$ numerically with nontrivial $\theta \approx 0.1207, 0.1875, 0.2380$ in $d = 1, 2$ and 3 , respectively.

In the light of our previous discussion, it is however easy to see why one would expect a nontrivial answer even in this simple case. Since the diffusion equation (3) is linear, the field $\phi(x, t)$ at a fixed point x as a function of t is clearly a Gaussian process with zero mean and is simply given by the solution of eq. (3), $\phi(x, t) = \int d^d x' G(\bar{x} - \bar{x}', t) \phi(x', 0)$, where $G(\bar{x}, t) = (4\pi t)^{-d/2} \exp[-x^2/4t]$ is the Green's function in d . Note that by solving eq. (3), we have already reduced the many-body diffusion problem to an 'effective' single-site Gaussian process in time t at fixed x . This therefore completes the first step of the two-step strategy mentioned earlier exactly. Now we turn to the second step, namely the 'no zero crossing' probability of this single-site Gaussian process. The two-time correlator of this can be easily computed from above and turns out to be non-stationary as in the examples specified by eqs (1) and (2). However by using the $T = \log(t)$ transformation as before, the normalized field reduces to a GSP in time T with correlator, $\langle X(T_1)X(T_2) \rangle = [\text{sech}(T/2)]^{d/2}$, where $T = |T_1 - T_2|$. Thus, once again, we are back to the general problem of the zero crossing of a GSP, this time with a correlator $f(T) = [\text{sech}(T/2)]^{d/2}$ which is very different from pure exponential form and hence is non Markovian. The persistence, $P_0(T)$ will still decay as $P_0(T) \sim \exp(-\theta T) \sim t^{-\theta}$ for large T (since $f(T)$ decays faster than $1/T$ for large T) but clearly with a nontrivial exponent.

Since persistence in all the examples that we have discussed so far (except the Ising model) reduces to the zero crossing probability of a GSP with correlator $f(T)$ [where $f(T)$ of course varies from problem to problem], let us now discuss some general properties of such a process. It turns out that a lot of information can already be inferred by examining the short-time properties of the correlator $f(T)$. In case of Brownian motion, we found $f(T) = \exp(-T/2) \sim 1 - T/2 + O(T^2)$ for small T . For the random acceleration problem, $f(T) = \frac{3}{2} \exp(-T/2) - \frac{1}{2} \exp(-3T/2) \sim 1 - 3T^2/8 + O(T^3)$ for small T and for the diffusion problem, $f(T) = [\text{sech}(T/2)]^{d/2} \sim 1 - \frac{d}{16} T^2 + O(T^3)$ as $T \rightarrow 0$. In general $f(T) = 1 - aT^\alpha + \dots$ for small T , where $0 < \alpha \leq 2$ (ref. 24). It turns out that processes for which $\alpha = 2$ are 'smooth' in the sense that the density of zero crossings ρ is finite, i.e. the number of zero crossings of the process in a given time T scales linearly with T . Indeed there exists an exact formula due to Rice²⁸, $\rho = \sqrt{-f''(0)}/\pi$ when $\alpha = 2$. However, for $\alpha < 2$, $f''(0)$ does not exist and this formula breaks down. It turns out that the density is infinite for $\alpha < 2$ and once the process crosses zero, it immediately crosses many times and then makes a long excursion before crossing the zero again. In other words, the zero's are not uniformly distributed over a given interval and in general the set of zeros has a fractal structure.²⁹

Let us first consider 'smooth' processes with $\alpha = 2$ such as random acceleration or the diffusion problem. It turns out that for such processes, one can make very good progress in calculating the persistence exponent θ .

The first approach consists of using an 'independent interval approximation' (IIA)⁵. Consider the 'effective' single-site process $\phi(T)$ as a function of the 'log-time' $T = \log(t)$. As a first step, one introduces the 'clipped' variable $\sigma = \text{sgn}(\phi)$, which changes sign at the zeros of $\phi(T)$. Given that $\phi(T)$ is a Gaussian process, it is easy to compute the correlator, $A(T) = \langle \sigma(0)\sigma(T) \rangle = \frac{2}{\pi} \sin^{-1}[f(T)]$, where $f(T)$ is the correlator of $\phi(T)$. Since the 'clipped' process $\sigma(T)$ can take values ± 1 only, one can express $A(T)$ as,

$$A(T) = \sum_{n=0}^{\infty} (-1)^n P_n(T), \quad (4)$$

where $P_n(T)$ is the probability that the interval T contains n zeros of $\phi(T)$. So far, there is no approximation. The strategy next is to use the following approximation,

$$P_n(T) = \langle T \rangle^{-1} \int_0^T dT_1 \int_{T_1}^{T_2} dT_2 \cdots \int_{T_{n-1}}^T dT_n \times Q(T_n) P(T_2 - T_1) \cdots P(T_n - T_{n-1}) Q(T - T_n), \quad (5)$$

where $P(T)$ is the distribution of intervals between two successive zeros and $Q(T)$ is the probability that an interval of size T to the right or left of a zero contains no further zeros. Clearly, $P(T) = -Q'(T)$. $\langle T \rangle = 1/\rho$ is the

mean interval size. We have made the IIA by writing the joint distribution of n successive zero-crossing intervals as the product of the distribution of single intervals. The rest is straightforward⁵. By taking the Laplace transform of the above equations, one finally obtains, $\tilde{P}(s) = [2 - F(s)]/F(s)$, where

$$F(s) = 1 + \frac{1}{2\rho} s[1 - s\tilde{A}(s)], \quad (6)$$

where the Laplace transform $\tilde{A}(s)$ of $A(T)$ can be easily computed knowing $f(T)$. The expectation that the persistence, $P_0(T)$ and hence the interval distribution, $P(T) \sim \exp(-\theta T)$ for large T , suggests a simple pole in the $\tilde{P}(s)$ at $s = -\theta$. The exponent θ is therefore given by the first zero on the negative s axis of the function,

$$F(s) = 1 + \frac{1}{2\rho} s \left\{ 1 - \frac{2s}{\pi} \int_0^\infty dT \exp(-sT) \sin^{-1}[f(T)] \right\}. \quad (7)$$

For the diffusion equation, $f(T) = [\text{sech}(T/2)]^{d/2}$ and $\rho = \sqrt{d/8\pi^2}$. We then get the IIA estimates of $\theta = 0.1203, 0.1862$ and 0.2358 in $d = 1, 2$ and 3 respectively, which should be compared with the simulation values, $0.1207 \pm 0.0005, 0.1875 \pm 0.0010$ and 0.2380 ± 0.0015 . For the random acceleration problem, $f(T) = \frac{3}{2} \exp(-T/2) - \frac{1}{2} \exp(-3T/2)$ and $\rho = \sqrt{3}/2\pi$ and we get, $\theta_{\text{ia}} = 0.2647$ which can be compared with its exact value, $\theta = 1/4$.

Though the IIA approach produces excellent results when compared to numerical simulations, it cannot however be systematically improved. For this purpose, we turn to the 'series expansion' approach⁶ which can be improved systematically order by order. The idea is to consider the generating function,

$$P(p, t) = \sum_{n=0}^{\infty} p^n P_n(t), \quad (8)$$

where $P_n(t)$ is the probability of n zero crossings in time t of the 'effective' single-site process. For $p = 0$, $P(0, t)$ is the usual persistence, decaying as $t^{-\theta(0)}$ as usual. Note that we have used $\theta(0)$ instead of the usual notation θ , because it turns out⁶ that for general p , $P(p, t) \sim t^{-\theta(p)}$ for large t , where $\theta(p)$ depends continuously on p for 'smooth' Gaussian processes. This has been checked numerically as well as within IIA approach⁶. Note that for $p = 1$, $P(1, t) = 1$ implying $\theta(1) = 0$. For smooth Gaussian processes, one can then derive an exact series expansion of $\theta(p)$ near $p = 1$. Writing $p^n = \exp(n \log p)$ and expanding the exponential, we then obtain an expansion in terms of moments of n , the number of zero crossings,

$$\log P(p, t) = \sum_{r=1}^{\infty} \frac{(\log p)^r}{r!} \langle n^r \rangle_c, \quad (9)$$

where $\langle n^r \rangle_c$ are the cumulants of the moments. Using $p = 1 - \varepsilon$, we express the right hand side as a series in powers of ε . Fortunately, the computation of the moments of n is relatively straightforward, though tedious for higher moments. We have already mentioned the result of Rice for the first moment. The second moment $\langle n^2 \rangle$ was computed by Bendat³⁰. We have computed the third moment as well⁶. For example, for 2-D diffusion equation, we get the series,

$$\theta(p=1-\varepsilon) = \frac{1}{2\pi} \varepsilon + \left(\frac{1}{\pi^2} - \frac{1}{4\pi} \right) \varepsilon^2 + O(\varepsilon^3). \quad (10)$$

Keeping terms up to second order and putting $\varepsilon = 1$ (in the same spirit as ε expansion in critical phenomena) gives, $\theta(0) = (\pi + 4)/4\pi^2 = 0.180899\dots$, just 3.5% below the simulation value, $\theta_{\text{sim}} = 0.1875 \pm 0.001$. Thus, this gives us a systematic series expansion approach for calculating the persistence exponent for any smooth Gaussian process.

Note that both the above approaches (IIA and series expansion) are valid only for 'smooth' Gaussian processes ($\alpha = 2$) with finite density ρ of zero crossings. What about the nonsmooth processes where $0 < \alpha < 2$, where such approaches fail? Even the Markov process, for which $f(T) = \exp(-\lambda T)$ is a non-smooth process with $\alpha = 1$. However, for the Markov case, one knows that the persistence exponent $\theta = \lambda$ exactly. One expects therefore that for Gaussian processes which may be non-smooth but 'close' to a Markov process, it may be possible to compute θ by perturbing around the Markov result.

In order to achieve this, we note that the persistence $P_0(T)$ in stationary time T , can be written formally⁴ as the ratio of two path integrals,

$$P_0(T) = \frac{2 \int_{\phi>0} \mathcal{D}\phi(\tau) \exp[-S]}{\int \mathcal{D}\phi(\tau) \exp[-S]} = \frac{Z_1}{Z_0}, \quad (11)$$

where Z_1 denotes the total weight of all paths which never crossed zero, i.e. paths restricted to either positive or negative (which accounts for the factor 2) side of $\phi = 0$ and Z_0 denotes the weight of all paths completely unrestricted. Here $S = \frac{1}{2} \int_0^T \int_0^T \phi(\tau_1) G(\tau_1 - \tau_2) \phi(\tau_2) d\tau_1 d\tau_2$ is the 'action' with $G(\tau_1 - \tau_2)$ being the inverse matrix of the Gaussian correlator $f(\tau_1 - \tau_2)$. Since $P_0(T)$ is expected to decay as $\exp(-\theta T)$ for large T , we get,

$$\theta = - \lim_{T \rightarrow \infty} \frac{1}{T} \log P_0(T). \quad (12)$$

If we now interpret the time T as inverse temperature β , then $\theta = E_1 - E_0$, where E_1 and E_0 are respectively the ground states of two 'quantum' problems, one with a 'hard' wall at the origin and the other without the wall.

For concreteness, first consider the Markov process, $f(T) = \exp(-\lambda |T|)$. In this case, it is easy to see that S is the action of a harmonic oscillator with frequency λ . The ground state energy, $E_0 = \lambda/2$ for an unrestricted oscillator with frequency λ . Whereas, for an oscillator with a 'hard' wall at the origin, it is well known that $E_1 = 3\lambda/2$. This then reproduces the Markovian result, $\theta = E_1 - E_0 = \lambda$. For processes close to Markov process, such that $f(T) = \exp(-\lambda T) + \varepsilon f_1(T)$, where ε is small, it is then straightforward to carry out a perturbation expansion around the harmonic oscillator action in orders of ε (ref. 4). The exponent θ , to order ε , can be expressed as,

$$\theta = \lambda \left(1 - \varepsilon \frac{2\lambda}{\pi} \int_0^\infty f_1(T) [1 - \exp(-2\lambda T)]^{-3/2} dT \right). \quad (13)$$

At this point, we go back momentarily to the zero temperature Glauber dynamics of Ising model. Note that the spin at a site in the Ising model takes values either 1 or -1 at any given time. Therefore, one really cannot consider the single-site process $s(t)$ as a Gaussian process. However, one can make a useful approximation in order to make contact with the Gaussian processes discussed so far. This is achieved by the so-called Gaussian closure approximation, first used by Mazenko³¹ in the context of phase ordering kinetics. The idea is to write, $s(t) = \text{sgn}[\phi(t)]$, where $\phi(t)$ now is assumed to be Gaussian. This is clearly an approximation. However, for phase ordering kinetics with nonconserved order parameter, this approximation has been quite accurate³¹. Note that within this approximation, the persistence or no flipping probability of the Ising spin $s(t)$ is same as the no zero crossing probability of the underlying Gaussian process $\phi(t)$. Assuming $\phi(t)$ to be a Gaussian process, one can compute its two-point non-stationary correlator self-consistently. Then, using the same 'log-time' transformation (with $T = \log(t)$) mentioned earlier, one can evaluate the corresponding stationary correlator $f(T)$. We are thus back to the general problem of zero crossing of a GSP even for the Ising case, though only approximately.

In 1-D, the correlator $f(T)$ of the underlying process can be computed exactly, $f(T) = \sqrt{2/(1 + \exp(2|T|))}$ (ref. 4) and in higher dimensions, it can be obtained numerically as the solution of a closed differential equation. By expanding around, $T = 0$, we find that in all dimensions, $\alpha = 1$ and hence they represent non-smooth processes with infinite density of zero crossings. Hence, we cannot use IIA or series expansion result for θ . Also due to the lack of a small parameter, we cannot think of this process as 'close' to a Markov process and hence cannot use the perturbation result. However, since $\theta = E_1 - E_0$ quite generally and since $\alpha = 1$, we can use a variational approximation to estimate E_1 and E_0 . We use as trial Hamiltonian that of a harmonic oscillator whose frequency λ is our tunable variational parameter⁴. We just

mention the results here, the details can be found in Majumdar and Sire, and Sire *et al.*⁴. For example, in $d = 1$, we find $\theta \approx 0.35$ compared to the exact result $\theta = 3/8$. In $d = 2$ and 3, we find $\theta \approx 0.195$ and 0.156. The exponent in 2-D has recently been measured experimentally⁹ in a liquid crystal system which has an effective Glauber dynamics and is in good agreement with our variational prediction.

So far we have been discussing about the persistence of a single spin in the Ising model¹¹. This can be immediately generalized to the persistence of 'global' order parameter in the Ising model. For example, what is the probability that the total magnetization (sum of all the spins) does not change sign up to time t in the Ising model? It turns out that when quenched to zero temperature, this probability also decays as a power law $\sim t^{-\theta_g}$ with an exponent θ_g that is different from the single spin persistence exponent θ . For example, in 1-D, $\theta_g = 1/4$ exactly¹¹ as opposed to $\theta = 3/8$ (ref. 2). A natural interpolation between the local and global persistence can be established via introducing the idea of 'block' persistence. The 'block' persistence is the probability $p_l(t)$ that a block of size l does not flip up to time t . As l increases from 0 to ∞ , the exponent crosses over from its 'local' value θ to its 'global' value θ_g .

When quenched to the critical temperature T_c of the Ising model, the local persistence decays exponentially with time due to the flips induced by thermal fluctuations but the 'global' persistence still decays algebraically, $\sim t^{-\theta_c}$, where the exponent θ_c is a new non-equilibrium critical exponent¹¹. It has been computed in mean field theory, in the $n \rightarrow \infty$ limit of the $O(n)$ model, to first order in $\varepsilon = 4 - d$ expansion¹¹. Recently this epsilon expansion has been carried out to order ε^2 (ref. 12).

Recently, the persistence of a single spin has also been generalized to persistence of 'patterns' in the zero temperature dynamics of 1-D Ising or more generally q -state Potts model. For example, the survival probability of a given 'domain' was found to decay algebraically in time as $\sim t^{-\theta_d}$ (ref. 14), where the q -dependent exponent $\theta_d(2) \approx 0.126$ (ref. 14) for $q = 2$ (Ising case), different from $\theta = 3/8$ and $\theta_0 = 1/4$. Also, the probability that a 'domain' wall has not encountered any other domain wall up to time t was found to decay as $\sim t^{-\theta_1}$ with yet another new exponent $\theta_1(q)$, where $\theta_1(2) = 1/2$ and $\theta_1(3) \approx 0.72$ (ref. 15). Thus, it seems that there is a whole hierarchy of nontrivial exponents associated with the decay of persistence of different patterns in phase ordering systems.

Another direction of generalization has been to investigate the 'residence time' distribution, whose limiting behaviour determines the persistence exponent³². Consider the effective single-site stochastic process $\phi(t)$ discussed in this paper. Let $r(t)$ denote the fraction of time the process $\phi(t)$ is positive (or negative) within time window $[0, t]$. The distribution $f(r, t)$ of the random variable r is the residence time distribution. In the limits

$r \rightarrow 0$ or $r \rightarrow 1$, this distribution is proportional to usual persistence. However, the full function $f(r, t)$ obviously gives more detailed information about the process than its limiting behaviours. This quantity has been studied extensively for diffusion equation^{32,33}, Ising model³⁴, Le'vy processes³⁵, interface models²⁰ and generalized Gaussian Markov processes³⁶.

The various persistence probabilities in pure systems have recently been generalized to systems with disorder¹⁷. For example, what is the probability that a random walker in a random environment (such as in Sinai model) does not cross the origin? Analytical predictions for the persistence in disordered environment have been made recently based on an asymptotically exact renormalization group approach¹⁷.

Another important application of some of these persistence ideas, experimentally somewhat more relevant perhaps, is in the area of interface fluctuations^{18,19}. The persistence in Gaussian interfaces such as the Edwards-Wilkinson model, the problem can again be mapped to a general GSP but with a non-Markovian correlator¹⁸. In this case, several upper and lower bounds have been obtained analytically¹⁸. For nonlinear interfaces of KPZ types, one has to mostly resort to numerical means¹⁹. The study of history dependence via persistence has provided some deeper insights into the problems of interface fluctuations^{19,20}.

On the experimental side, the persistence exponent has been measured in systems with breath figures⁸, soap bubbles¹⁰ and twisted nematic liquid crystal exhibiting planar Glauber dynamics⁹. It has also been noted recently³⁷ that persistence exponent for diffusion equation may possibly be measured in dense spin-polarized noble gases (Helium-3 and Xenon-129) using NMR spectroscopy and imaging³⁸. In these systems, the polarization acts like a diffusing field. With some modifications these systems may possibly also be used to measure the persistence of 'patterns' discussed in this paper.

In conclusion, persistence is an interesting and challenging problem with many applications in the area of nonequilibrium statistical physics. Some aspects of the problem have been understood recently as reviewed here. But there still exist many questions and emerging new directions open to more theoretical and experimental efforts.

1. Derrida, B., Bray, A. J. and Godrèche, C., *J. Phys. A*, 1994, **27**, L357; Stauffer, D., *J. Phys. A*, 1994, **27**, 5029; Krapivsky, P. L., Ben-Naim, E. and Redner, S., *Phys. Rev. E*, 1994, **50**, 2474.
2. Derrida, B., Hakim, V. and Pasquier, V., *Phys. Rev. Lett.*, 1995, **75**, 751 and *J. Stat. Phys.*, 1996, **85**, 763.
3. Bray, A. J., Derrida, B., and Godrèche, C., *Europhys. Lett.*, 1994, **27**, 175.
4. Majumdar, S. N. and Sire, C., *Phys. Rev. Lett.*, 1996, **77**, 1420; Sire, C., Majumdar, S. N. and Rudinger, A., *Phys. Rev. E*, cond-mat/9810136, (to appear).

5. Majumdar, S. N., Sire, C., Bray, A. J. and Cornell, S. J., *Phys. Rev. Lett.*, 1996, **77**, 2867; Derrida, B., Hakim, V. and Zeitak, R., *ibid.*, 2871.
6. Majumdar, S. N. and Bray, A. J., *Phys. Rev. Lett.*, 1998, **81**, 2626.
7. Watson, A., *Science*, 1996, **274**, 919.
8. Marcos-Martin, M., Beysens, D., Bouchaud, J-P, Godrèche, C. and Yekutieli, I., *Physica*, 1995, **D214**, 396.
9. Yurke, B., Pargellis, A. N., Majumdar, S. N. and Sire, C., *Phys. Rev. E*, 1997, **56**, R40.
10. Tam, W. Y., Zeitak, R., Szeto, K. Y. and Stavans, J., *Phys. Rev. Lett.*, 1997, **78**, 1588.
11. Majumdar, S. N., Bray, A. J., Cornell, S. J. and Sire, C., *Phys. Rev. Lett.*, 1996, **77**, 3704.
12. Oerding, K., Cornell, S. J. and Bray, A. J., *Phys. Rev. E*, 1997, **56**, R25; Oerding, K. and van Wijland, F., *J. Phys. A*, 1998, **31**, 7011.
13. Cueille, S. and Sire, C., *J. Phys. A*, 1997, **30**, L791; *Euro. Phys. J. B*, 1999, **7**, 111.
14. Krapivsky, P. L. and Ben-Naim, E., *Phys. Rev. E*, 1998, **56**, 3788.
15. Majumdar, S. N. and Cornell, S. J., *Phys. Rev. E*, 1998, **57**, 3757.
16. Cardy, J., *J. Phys. A*, 1995, **28**, L19; Ben-Naim, E., *Phys. Rev. E* 1996, **53**, 1566; Howard, M., *J. Phys. A*, 1996, **29**, 3437 Monthus, C., *Phys. Rev. E*, 1996, **54**, 4844.
17. Fisher, D. S., Le Doussal, P. and Monthus, C., *Phys. Rev. Lett.* 1998, **80**, 3539; cond-mat/9811300; Le Doussal, P. and Monthus C., cond-mat/9901306, (to appear).
18. Krug, J., Kallabis, H., Majumdar, S. N., Cornell, S. J., Bray, A. J. and Sire, C., *Phys. Rev. E*, 1997, **56**, 2702.
19. Kallabis, H. and Krug, J., cond-mat/9809241, (to appear).
20. Toroczkai, Z., Newman, T. J. and Das Sarma, S., cond-mat/9810359, (to appear).
21. Frachebourg, L., Krapivsky, P. L. and Ben-Naim, E., *Phys. Rev. Lett.*, 1996, **77**, 2125; *Phys. Rev. E*, 1996, **54**, 6186.
22. Swift, M. R. and Bray, A. J., cond-mat/9811422, (to appear).
23. Feller, W., *Introduction to Probability Theory and its Applications*, Wiley, New York, 3rd edn, 1968, vol. 1.
24. Slepian, D., *Bell Syst. Tech. J.*, 1962, **41**, 463.
25. Wang, M. C. and Uhlenbeck, G. E., *Rev. Mod. Phys.*, 1945, **17**, 323.
26. Sinai, Y. G., *Theor. Math. Phys.*, 1992, **90**, 219.
27. Burkhardt, T. W., *J. Phys. A*, 1993, **26**, L1157.
28. Rice, S. O., *Bell Syst. Tech. J.*, 1944, **23**, 282; 1945, **24**, 46.
29. Kac, M., *SIAM Rev.*, 1962, **4**, 1; Blake, I. F. and Lindsey, W. C., *IEEE Trans. Inf. Theory*, 1973, **19**, 295.
30. Bendat J. S., *Principles and Applications of Random Noise Theory*, Wiley, New York, 1958.
31. Mazenko, G. F., *Phys. Rev. Lett.*, 1989, **63**, 1605.
32. Dornic, I. and Godrèche, C., *J. Phys. A*, 1998, **31**, 5413.
33. Newman, T. J. and Toroczkai, Z., *Phys. Rev. E*, 1998, **58**, R2685.
34. Drouffe, J-M. and Godrèche, C., cond-mat/9808153, (to appear).
35. Baldassarri, A., Bouchaud, J. P., Dornic, I. and Godrèche, C., cond-mat/9805212, (to appear).
36. Dhar, A. and Majumdar, S. N., *Phys. Rev. E*, cond-mat/9902004, (to appear).
37. Walsworth, R. L. (private communication).
38. Tseng, C. H., Peled, S., Nascimben, L., Oteiza, E., Walsworth, R. L. and Jolesz, F. A., *J. Magn. Reson., Ser. B*, (to appear).

ACKNOWLEDGEMENTS. I thank my collaborators C. Sire, A. J. Bray, S. J. Cornell, J. Krug, H. Kallabis, B. Yurke, A. Pargellis and A. Dhar. I also thank D. Dhar for many valuable suggestions and discussions. I am grateful to M. Barma, B. Derrida and C. Godrèche for useful discussions and to CNRS, Université' Paul Sabatier for hospitality where the whole series of work began.