Differential and Integral Operators. I. Gohberg, R. Mennnicken and C. Tretter (eds). Birkhauser Verlag, PO Box 133, CH4010, Basel, Switzerland, 1998. 324 pp. Price: DM 178, SFr 148.

This volume is the proceedings of a conference on operator theory and its applications and the main theme is the spectral theory of linear and nonlinear problems. The composition is articles in the theory of Sturm–Liouville problems, Wiener–Hopf factorization, analysis on Krein spaces and spectral problems of self-adjoint operators. I will concentrate on the last item in this review and discuss work relating to four papers in the volume.

One of the questions concerns the three-body problem of Quantum Mechanics and the paper of S. Albeverio and K. A. Makarov addresses the spectral theory of singular perturbations in this context. The main questions in this theory have been the spectral nature of the three-body operators and, in particular, absence of singular spectrum, the spectral equivalence of the absolutely continuous spectrum to that of certain subclasses of operators (asymptotic completeness) and the nature of the point spectrum. The first two questions have been the study for a long time in the theory and in the eighties the problems have been essentially settled for most reasonable operators. The pathological behaviour appears for singular perturbations of the type addressed in this paper and it deals with the accumulation points of the discrete spectrum of the three-body operator. The bottom of the essential spectrum being one such accumulation point is known as the 'Effimov effect'. The paper of the authors is a contribution to that study.

In the case of one dimension, there are Schrödinger operators which are the self-adjoint operators associated with the Sturm–Liouville differential expressions (of the type $d/d(p(x))(d/dx) + q(x)$, plus possibly some boundary conditions) whose spectral theory and inverse spectral theory is an ongoing study for several decades now. The original complete inverse spectral theory is given by Gelfand and B. M. Levitan which recovers a given second order differential expression (of a certain type) from the spectral function. The paper of L. A. Sakhnovich extends this theory (in the form given by M. G. Krein) to differential expressions with matrix coefficients.

In the above context, the study of differential operators with periodic coefficients relies on the Floquet theory, where one considers the fundamental solution of an equivalent first order, but matrix valued equation $y'(x) = A(x)y(x)$, of dimension n (when the order of the differential equation is n). If the coefficients are of period T, then the fundamental solution $y(T)$ contains all the necessary spectral information. The Floquet exponents are obtained via a regularized determinant of a Hilbert–Schmidt operator associated with the problem.

Finally the paper of R. Weikard is an analysis of a special differential expression $-(d^2/d^2) + q(x)$, $(q(x) = (a/x^2))$, the motivation being to study the finer properties of the differential expression *vis-à-vis* the spectrum of the associated self-adjoint operator(s). In particular, the study of the nature of $q(x)$ associated with a given spectrum is a hard problem in this area and such operators provide a solution to studying the spectra associated with elliptic functions $q(x)$ (q need not be real valued).

Most of the articles in the volume will probably appear elsewhere in journals, so this volume should receive in my opinion a low priority for purchase.

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Laws of Chaos – Invariant Measures and Dynamical Systems in One Dimension, Abraham Boyarsky and P. Gora. Birkhauser Verlag AG, PO Box 133, CH-4010, Basel, Switzerland. 399 pp. Price: SFr 108.

The question of existence of a finite invariant measure in the context of dynamical systems was discussed by G. D. Birkhoff and Paul Smith in their expository paper[1] and we will first review their discussion with only very minor changes.

Let σ be a Borel automorphism on a standard Borel space $(X, \mathscr{B})$ which preserves the class of null sets of a probability measure m on $\mathscr{B}$. We begin by introducing a function ϕ on $\mathscr{B}$ defined as follows: $E \in \mathscr{B}$ be divided into finite number of mutually disjoint measurable sets E_i, $E = \cup E_i$, $E_i \cap E_j = \varnothing$ for $i \neq j$. Then $\phi(E)$ is the lower bound of the finite sums $\sum m(\sigma^{n_i} E_i)$ with respect to all possible methods of subdivisions of E into finite number of measurable sets, and all possible choices of integers n_i.

It is clear that ϕ may be identically zero in which case we say that 'X is compressible into a set of arbitrarily small measure in the sense of Birkhoff and Smith'. This can happen, for example, when X is a sphere and σ is an invertible analytic transformation of a sphere such that each circle parallel to the equator closes down on north (or south) pole with indefinite iteration of σ (or σ^{-1}). On the other hand, for an automorphism which preserves the measure, $\phi(E) = m(E)$, for all measurable sets E. In any case it follows immediately that

$$\phi(E) \leq m(E) \text{ for all } E \in \mathscr{B}.$$

If E_n, $n \in N$, is a sequence of sets in $\mathscr{B}$ decreasing to the empty set then $\phi(E_n)$ decreases to zero since $m(E_n)$, being countably additive, decreases to zero. Since ϕ can be shown to be finitely additive it follows it is σ-invariant countably additive and absolutely continuous with respect to m, $\phi(X) = 1$.

On the suggestion of Birkhoff the question of existence of a finite invariant measure was further studied by E. Hopf who modified the notion of compressibility of Birkhoff and Smith by permitting countable partitions of the space X.

Call two sets $A, B \in \mathscr{B}$ equivalent by countable decomposition if we can (i) write A as a countable union of pairwise disjoint sets $A_i \in \mathscr{B}$, $i \in N$, (ii) write B as a countable union of pairwise disjoint sets $B_i \in \mathscr{B}$, $i \in N$, (iii) find integers n_i, $i \in N$, such that for each i, $\sigma^{n_i} A_i = B_i$.

If A and B are equivalent by countable decomposition then we write $A \sim B$, and say that B is a copy of A. It can be verified that the relation $\sim$ on $\mathscr{B}$ is indeed an equivalence relation. If $A \sim B$ then A and B have the same measure with respect any σ-finite measure invariant under σ. If A is a pairwise disjoint union of measurable sets A_i, $i \in N$, and B is a pairwise disjoint union of measurable sets B_i, $i \in N$, and if $A_i \sim B_i$ for each i, then $A \sim B$. We say that A and B are equivalent by countable decomposition (modulo m) if there exist sets M and N in $\mathscr{B}$ of measure zero such that $A \Delta M$ and $B \Delta N$ are equivalent by countable decomposition. A set $A \in \mathscr{B}$ is said to be compressible in the sense of Hopf if there exists $B \subseteq A$ such that $A \sim B$ and $m(A - B) > 0$. Clearly the notion of compressibility in the sense of Hopf depends only on the σ-ideal of m-null sets in $\mathscr{B}$ and not on m itself. If μ is a finite measure on $\mathscr{B}$ invariant under σ and having the same null sets as m, then $A \sim B \pmod m$ implies that $\mu(A) = \mu(B)$. Moreover, whenever such a μ exists, no measurable set of positive measure can be compressible in the sense of Hopf. In particular, X cannot be compressible in the sense of Hopf whenever such a μ exists. E. Hopf[2] proved the difficult converse of this, namely, that *if X is not compressible in the sense of Hopf then there exists a finite measure μ invariant under σ and having the same null sets as m.*

In ref. 3, Hajian and Kakutani gave rather simple necessary and sufficient condition for a non-singular automorphism to admit an invariant probability measure absolutely continuous with respect to the given one. Call a measurable subset A of X weakly wandering if the sets $\sigma^n(A)$ are pair-wise disjoint as n runs over some infinite subset of positive integers. Hajian and Kakutani show that *σ admits an invariant probability measure if and only if there is no weakly wandering set of positive measure.*

The book under review deals with this problem at a much more concrete level for some special classes transformations (not necessarily invertible) on intervals of the real line. The results mentioned above, being of foundational nature, do not seem to be immediately applicable and different methods have to be devised to exhibit the existence of a finite invariant measure, absolutely continuous with respect to a given one. Since transformations considered are much more concrete (piece-wise smooth maps on intervals) interesting calculations combined with some functional analytic considerations permit one to exhibit finite invariant measure for a given transformation.

Consider a measurable map τ: $[0, 1] \to [0, 1]$ such that $\lambda \circ \tau^{-1}$ is absolutely continuous with respect to λ, the Lebesgue measure on $[0, 1]$. For $f \in L^1([0, 1], \lambda)$, the measure

$$\mu(A) = \int_{\tau^{-1}(A)} f d\lambda, \ A \subset [0, 1]$$

is absolutely continuous with respect to λ. The map P_τ: $f \to (d\mu/d\lambda)$, called Frobenius-Perron operator and referred to as the hero of the book, is positive, and a linear contraction. Further, $P_{\tau \circ \sigma} = P_\tau \circ P_\sigma$. These and other properties of P_τ are presented in chapter 3. With the aid of these properties the rest of the book embarks on a full scale discussion of dynamical and statistical properties of those τ which belong to one or more of the following or similar classes:

(i) The class $\mathscr{T}$ of those τ which (a) are piece-wise smooth and expanding in the sense that $|\tau'_i| > \alpha > 1$, where τ_i is the restriction of τ to the interval I_i where it is smooth, (b) the function $g(x) = 1/|\tau'(x)|$, (at the end point of an interval of the partition associated to τ, g is defined to be one of the one-sided derivatives which exists) is of bounded variation.

(ii) The class $\mathscr{T}_{pc}$ of piece-wise continuous and convex functions, $\tau(a_{i-1}) = 0$, $\tau'(a_{i-1}) > 0$, $\tau'(0) > 1$.

(iii) Class $\mathscr{T}_M$ of piece-wise smooth maps which are homeomorphisms of each subintervals of a partition onto a union of subintervals of the same partition. Such a τ is called Markov, in addition if these homeomorphisms are linear then we call τ a piece-wise linear Markov transformation.

It is shown, following Lasoto and Yorke (but sometimes with different proofs) that in case (i) and (ii) the τ admits an invariant absolutely continuous (w.r.t. λ) probability measure. In case (i) the theorem is rather tight since, as an example of Lasoto and Yorke shows, the theorem fails if $|\tau'|$ is < 1 even at one point. The number of absolutely continuous invariant measures is finite, and the spectrum of τ consists of finite number of eigenvalues together continuous spectrum which is of Lebesgue type. A suitable power of τ, restricted to a subset, is indeed exact. In case τ is uniquely ergodic and weakly mixing then it is a Bernouli shift, hence amenable to questions concerning central limit theorem and the law of large numbers. These are discussed as also the form and the support of invariant densities. The Markov transformations and piece-wise linear Markov transformations are amenable to matrix analysis and this is taken up in Chapter 9. The stability of invariant measure under perturbations, random or non-random, is discussed. The inverse problem, namely, of computing τ given the invariant density f is treated and the last chapter considers some applications of the existence of invariant probability measures including Kolodziej's proof of Poncelet's theorem in projective geometry.

There is a good collection of illustrative examples and also a number of exercises for each chapter. The book supplements the existing texts on ergodic theory and applied dynamics by fully discussing the dynamical properties of piece-wise smooth τ on an interval of the real line and so recommended for graduate students and more advanced scientists interested in these fields.

1. Birkhoff, G. D. and Smith, P. A., *J. Math.*, 1928, Tome VII-Fasc.IV, 345–379; *Birkhoff Collected Mathematical Papers*, 1968, vol. 2, pp. 360–394.
2. Hopf, E., *Trans. Am. Math. Soc.*, 1932, **34**, 373–393.
3. Hajian, A. B. and Kakutani, S., *Trans. Am. Math. Soc.*, 1964, **110**, 136–151.

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Diamonds in India. T. M. Babu, Geological Society of India, PB No. 1922, Gavipuram, Bangalore 560 019, India. 1998. 331 pp. Price: Rs 300.

The very mention of diamond evokes fantasies of fabulous riches. Up to the Middle Ages they were so rare and expensive that only powerful potentiates were able to afford diamonds. But in modern times even ordinary people are able to possess a few, thanks to the discovery of numerous diamond deposits on all the continents of the Earth and