Harish-Chandra and his mathematical legacy: Some personal recollections

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Harish-Chandra was the greatest mathematician of Indian origin in our times. He was an algebraist and an analyst who was the principal creator of infinite-dimensional representation theory, a subject which through his work and many others has become a central topic in contemporary mathematics.

While his work is largely cumulative and thus has a certain monumental quality, it has proved a fertile ground for progress in several fields of mathematics.

Harish-Chandra was born on 11 October 1923 in Kanpur in North India. His father, Chandrakishore, was a Civil Engineer who at the end of his career was Executive Engineer of the Uttar Pradesh Irrigation Works. Much of Harish-Chandra's childhood was spent at the home of his maternal grandfather in Kanpur. This was also home to a number of relatives, and apparently Harish-Chandra did not always find the turbulent atmosphere congenial. However, he derived from there a love of music, both the ragas from his homeland and later also music of western composers.

His early education was given special attention, a tutor was hired, and in school he was younger than his classmates. He finished intermediate college in Kanpur at the age of 16, obtained his Master's Degree at the University of Allahabad in 1943 at the age of 20. On 10 October 1993 (when he would have been 70) a bust of Harish-Chandra was unveiled at the Mehta Institute in memory of their brilliant student.

In 1940 he came across Dirac's classic, *Principles of Quantum Mechanics* and this aroused his interest in theoretical physics. He then came to Bangalore and joined the Physics Department, Indian Institute of Science and decided (on C. V. Raman's advice) to work with Homi Bhabha, then a Reader in the Physics Department. K. S. Krishnan too took special interest in Harish-Chandra and recommended him as a research student to Homi Bhabha. Already by 1944 he and Bhabha published a joint paper 'On the Theory of Point Particles'.

The image of young Harish-Chandra, as conveyed by Langlands through recollections of Harish-Chandra's mother-in-law is that of a young man, precocious in his studies, somewhat high strung, yet timid and frequently ill.

The mild-mannered and gentle Krishnan inspired in Harish-Chandra great respect and affection. But as Langland puts it, 'Harish-Chandra's ascetic nature did not allow him to perceive the virtues accompanying the high-living Bhabha's extravaganza'.

The first papers of Harish-Chandra were published in the *Proc. Royal Society, London*, and came to the attention of Dirac. Probably at the recommendation of Bhabha, Dirac accepted Harish-Chandra as a research student in Cambridge. In 1944, Dirac had written one of the first papers on the infinite-dimensional representations of the Lorentz group. The classification of all such representations became the topic of Harish-Chandra's thesis (1947). This classification was carried out at about the same time by Bargmann (USA) and Gelfand-Naimark in the Soviet Union.

In 1947–48 Dirac went to the Institute for Advanced Study in Princeton and Harish-Chandra was appointed as his assistant. But at that time his interest shifted to mathematics. He realized that some of the arguments in his thesis were shaky and not sufficiently rigorous. Mentioning this to Dirac, the latter responded 'I am not interested in proofs, but only in what nature does'. Harish-Chandra stated later, 'This remark confirmed my growing conviction that I did not have the mysterious sixth sense which one needs in order to succeed in physics and I soon moved over to mathematics'. We mathematicians can be truly grateful for that.

It is somewhat amusing that in the introduction to his thesis he expresses a mild disagreement with some of Dirac's physical interpretations.

Dirac apparently approved heartily of Harish-Chandra's shift in fields and during this year at the Institute he plunged into mathematics with tremendous appetite, eager to master it all. However, because of his own background and since Chevalley's book on Lie groups had recently appeared he turned to a systematic study of this subject, undoubtedly stimulated by the presence in Princeton of Hermann Weyl, Claude Chevalley, and his contemporary George D. Mostow, who had recently completed his thesis in this field.

Considering his rather irregular mathematical background he seems to have devoured the existing Lie group theory, tough as it was, at a ferocious rate; in April 1948 he submitted to the *Annals of Mathematics* a new algebraic proof of one of the major theorems in the subject, Ado's theorem. This theorem can be briefly explained as follows.

A Lie algebra is a vector space V, $\dim V < \infty$ such that to each pair $X, Y \in V$ is associated a third vector $[X, Y] \in V$. This assignment is assumed to satisfy:

$$[X, Y] = -[Y, X]$$
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Example: $V = M_n = \{\text{all } n \times n \text{ matrices}\}$, where $[X, Y] = XY - YX$.

Ado's Theorem. *Every Lie algebra is isomorphic to a subalgebra of some* M_n.

During the spring of 1948, Harish-Chandra wrote 5 other papers on Lie theory. Some gave new and better proofs of known results. Around 1950, armed with a complete command of Lie group theory, Harish-Chandra embarked on a project which was to occupy him for the rest of his life, namely 'Infinite-dimensional representations of semisimple Lie groups'. Semisimple Lie groups have a very rich structure theory, due primarily to Élie Cartan, so this project was indeed a very natural continuation of his thesis.

In this undertaking he steered a middle course between the significant examples studied by Bargmann, Gelfand and Naimark and the general theories for locally compact groups developed by Godement, Mackey, Segal and others. This was also a very wise choice because here he could take full advantage of the rich structure theory of semisimple Lie groups. The resulting theory is infinitely richer than the corresponding results for general locally compact groups.

With hindsight this choice of topic seems almost automatic; so it is worth recalling that this choice went against the prevailing winds at the time (1950). Analysis on *abelian* locally compact groups G was then almost completely developed and there was a general optimism around that the same could be done for nonabelian locally compact groups by functional analysis tools: *Haar Measure* and the theory of *Operator Algebras*. But Harish-Chandra's work was soon to change that attitude.

Career

- M Sc 1943 – University of Allahabad
- 1943–1945 – Research student, Indian Institute of Science, Bangalore
- 1945–1947 – Research student, University of Cambridge
- Ph D 1947 – University of Cambridge.
- 1947–1948 – Institute for Advanced Study, Princeton
- 1948–1949 – Jewett Fellowship, Harvard
- 1950–1963 – Columbia University with the following leaves:
- 1952–1953 – Tata Institute, Bombay
- 1955–1956 – Institute for Advanced Study, Princeton
- 1957–1958 – Guggenheim Fellow, Paris
- 1961–1962 – Institute for Advanced Study, Princeton.
- 1963–1983 – Permanent member, Institute for Advanced Study, Princeton.

From this chronology it is clear that Harish-Chandra's career as a professor at a University spanned only 8 years and I do not believe he had any graduate students writing a thesis under his direction. However, he gave regularly very advanced graduate courses on fields, which he was keenly interested in (Siegel's theorem on quadratic forms, class field theory, etc.) but not until 1962–63 did he give a course related to his own work.

After he came to the Institute in Princeton in 1963, he gave regular lectures on work in progress. These lectures were not only attended by interested members at the Institute but also by occasional graduate students at Princeton University.

His pioneering work on representation theory started with his 1951 paper 'On some applications of the universal enveloping algebra of a semisimple Lie algebra' (*Transactions, AMS*).

This algebra, denoted B, can be constructed as follows. Let $X_1, ..., X_n$ be a basis of the Lie algebra $\mathfrak{g}$. Then B is the set of all noncommutative polynomials in $X_1, ..., X_n$, where however, $X_1X_2 - X_2X_1 = [X_1, X_2]$. This first paper is influenced by work of Chevalley and the algebra B is a central object in all of Harish-Chandra's later works. It is very important that if G is a Lie group with Lie algebra $\mathfrak{g}$ then B can be viewed as the algebra of left invariant differential operators on G. This paper was followed by three others in the *Transactions* (1953–1954). These papers created the foundation of the subject.

He now found himself in new unexplored territory and everywhere he looked there were natural, conceptually compelling problems to deal with. Looking backwards and remembering the primitive state of the subject in 1950, one does not know what to admire most: the uncanny insight in finding the right chain of intermediary results to prove, or the consummate skill in carrying out the complex proofs.

Using his tool of the so-called *analytic vectors*, he showed that irreducibility questions for unitary representations could be settled by purely algebraic criteria. This initiated the study of what is now called *Harish-Chandra modules*. If K is a maximal compact subgroup of G, Harish-Chandra proved that a given K-type occurs only finitely many times in a given irreducible representation of G and that there are only finitely many irreducible representations of G with a given infinitesimal character containing a given K-type.

Finally, inspired by work of Gelfand and Naimark on the complex classical groups, he defined the principal series of representations of G and showed (his sub-quotient theorem) that essentially all irreducible unitary representations of G can be captured in this way (although a bit indirectly).

These papers were primarily algebraic but they formed the basis of the theory of characters of representations. Gelfand and Naimark had shown for the complex classical groups that even for the infinite-dimensional representations of the principal series one can attach a character:

$$\chi\pi(g) = \sum_1^\infty (\pi(g)e_i, e_i)$$

because this defines a function on the set of all regular elements g in G. Harish-Chandra proceeded via distribution theory, showed that if π is any irreducible unitary representation the operator $\pi(f) = \int_G f(g)\pi(g)\mathrm{d}g$ (f being a smooth compactly supported function) is of trace class and that the functional $f \rightarrow Tr(\pi(f))$ is a distribution on G in the sense of Schwartz (the K-finiteness theorem, which I mentioned earlier, is the basis of this proof). This functional $\chi\pi$ is called the character of the representation and was intensely studied by Harish-Chandra in the early sixties. In a series of five profound papers, he showed that $\chi\pi$ is actually a locally integrable function on G, analytic on the dense regular set G', but may blow up on the lower dimensional set $G - G'$.

His principal aim was an explicit Plancherel formula

$$f(1) = \int_{\hat{G}} \chi\pi(f)\mathrm{d}\pi,$$

where $\hat{G}$ = set of equivalence classes of irreducible unitary representations of G and $\mathrm{d}\pi$ is a measure which should be explicitly given in terms of a concrete parametrization of $\hat{G}$.

Such a formula had been proved by Gelfand and Naimark for the complex classical groups already in the late forties. This did not cover the case of the group $SL(2, R)$ (the very simplest simple Lie group). This was done by Harish-Chandra in 1952 (based partly on Bargmann's paper already mentioned). Already this case shows that the case of real G is much more complicated than the case of complex G and this led Harish-Chandra to his very deep theories of *orbital integrals* and the *discrete series*. Each of these is a beautiful creation in its own right and enters significantly in the final Plancherel formula (with explicit $d\pi$). As another step along the way, he developed the theory of spherical functions in two beautiful papers in 1958 where the remarkable c-functions first appears.

His final paper completing the Plancherel formula was published in 1976.

Thus, Harish-Chandra's *magnum opus*, which one might call 'Harmonic Analysis on Semisimple Lie groups' stretched unbroken from 1951 to 1976. The cumulative nature of this work of 25 years leads one to characterize it as monumental; yet even this does not do justice to the courage and pioneering efforts, conceptual and technical, which are needed to overcome the formidable obstacles on the way.

Research highlights

1. The Algebraization of Represenation Theory (1951–54), K-finiteness, Harish-Chandra Modules.
2. The theory of Characters (1955–65). The local integrability theorem.
3. The Orbital Integral Theory (1957–66).
4. The Discrete Series (1963–66). Determination through the characters.
5. The Spherical Function Theory (1958). The introduction of the c-function.
6. The Plancherel Formula (1951–76).

A good deal of this work was done during periods of persistent ill health. It could hardly have been completed without the brave support of his dear wife Lily, who with patience and courage during serious illnesses remained a mainstay in his life.

I first met Harish-Chandra in 1954 when he was invited to Princeton to give a lecture on representation theory to the physicists. It is an indication of how little known the subject was then that I (who was a graduate student at the time) was, I believe, the only mathematician in the audience. While the subject was entirely foreign to me, the lecture made a lasting impression. It inspired my beginning systematic study of semisimple Lie groups and symmetric spaces. However, my tastes were more geometric than Harish-Chandra's and I found it quite rewarding to relate some of his work to Cartan's geometric development of his theory of symmetric spaces.

In 1959 I happened to visit him and Lily in their small apartment in Butler Hall near Columbia University. I had been reading recently his two 1958 papers on spherical functions (total about 130 pages). I still remember a tiny desk in that apartment near the kitchenette and found out in amazement that in this ungainly workspace he had actually written these long magnificent papers. This tiny desk always symbolizes for me the ascetic discipline in his work habits. I understand that he wrote these long technical papers in three months.

Of several honours Harish-Chandra received, I mention specifically the Cole Prize in 1954 and the Ramanujan Medal of the Indian Science Academy in 1974. During the fifties and sixties, the mathematical public watched in awe as dozens of major mathematical papers flowed from his pen; the subjects revealed a structure of surprising richness and beauty. As an example, one may mention the remarkable and mysterious c-function which is attached to any semisimple Lie group like an engraved jewel and has continued to reveal remarkable features. It is not surprising that as these mysteries unfolded Harish-Chandra became completely captivated by the subject. He often expressed the view (which is of course shared by many mathematicians) that many of the most exalted mathematical results reveal themselves to us as if they were creations of a superior culture or even of a divine force. According to this view, the role of the mathematician is to uncover these creations like an archealogist or to foresee them like a clairvoyant rather than to invent them with his own mind.

The year 1959–60 I spent at Columbia University and Harish-Chandra and I shared an office there. He was then giving a full year's course on Siegel's theorems for quadratic forms and I was giving a course on Lie groups. I remember with a certain nostalgia many long and leisurely discussions we had on this subject. Shortly before I left Columbia for MIT in the Spring of 1960, he called me up and told me that he had succeeded in proving the finiteness of the volume of the fundamental domain for arithmetic subgroups of semisimple Lie groups. This had been his ambition since his year in Paris 1957–58. He explained his proof to me with great pleasure. Knowing that he had not been altogether well that Spring, I remember telling Lily that same evening that this was a good occasion for her to persuade him to take a long vacation. But, ironically, that summer he became rather seriously depressed, perhaps as a result of overwork.

With long hours of intense work, Harish-Chandra was an extremely prolific writer. He pursued his goals with firm determination, avoiding unnecessary distractions. Since each of his papers was for the most part based on his own previous work, the writing was easy for him. For quite a while the frequency of these long papers caused more admiration than assimilation by the mathematical public. Determined readers of his papers were, however, pleased to find them written with extreme clarity and care. A patient reader found no insurmountable obstacles because tedious details were given their due attention and not left to the reader to verify. This benefitted this new subject of representation theory in two ways: 1) Workers in this field never hesitated using Harish-Chandra's results even if they had not worked through or even seen the proofs; 2). His uncompromising and precise style became an example for others to follow.

I am sure that this has significantly contributed to the vigorous health which representation theory has enjoyed in recent years. In fact, the principal results of Harish-Chandra's 'Harmonic Analysis on semisimple Lie Groups' are now in the process of receiving a significant generalization to semisimple symmetric spaces G/H. (This means that G is a semisimple Lie group and H is the fixed group of an involutive automorphism of G.) Since G can be written as $G \times G/DG$ (DG = diagonal in $G \times G$) and $G \times G$ has the involution $(x, y) \rightarrow (y, x)$ with fixed point group DG this is a very extensive generalization of Harish-Chandra's theory. The principal participants in this development are Flensted-Jensen, Oshima, Matsuki, Delorme, van den Ban and Schlichtkrull. Much of this research is still not published but an up-to date account can be found in an article by Ban, Flensted-Jensen and Schlichtkrull in 'Proceedings of the Edinburgh Conference on Automorphic Forms and Representation Theory', published recently by the American Mathematical Society. In contrast to Harish-Chandra's development, the orbital integral theory is not needed for the Plancherel formula but will clearly become an interesting subject in its own right, generalized to G/H. Many interesting results already exist.

The Institute in Princeton had few formal duties. There Harish-Chandra gave regular lectures on his own work. He delighted in describing his current ideas to others and it was indeed a pleasure to see his ideas emerging in their freshness. I remember a period in the mid-sixties when he was writing his major papers on the discrete series. I had an adjoining office at the Institute. Often I could not help hearing through the wall that he was loudly singing at his desk. Sometimes on such occasions I would knock on his door on a small pretext and find him delighted at the interruption and eager to explain to me what he was doing. I also remember him once coming to my office with a troubled look, having found an unexpected gap in a proof he was writing up. He came to ask me about some result I had proved relating bi-invariant differential operators on G to left invariant operators on G/K. It turned out that these results convinced him that the gap could be filled but it would also mean that the theorem had to be proved by classification, a method which he had always abhorred. Some of us at the Institute had a good time teasing him a little that now he would have to compromise on those principles. He took this with good humour but knowing now that the theorem was basically right he came quickly up with a general proof (without classification).

As I mentioned, during 1960 persistent illness began to cast an ominous shadow over Harish-CHandra's life. He also realized that the long bouts of intensive mathematical activity might be harmful. On the other hand, mathematical meditation had become such an integral part of his life that abstaining from it seemed unnatural and strenuous. There was a compulsive streak in him which he could not easily control, especially if he found an unexpected obstacle when writing up a proof.

However, through expert medical treatment and Lily's devoted care a certain equilibrium developed. Thus I remember that the years 1964–66, when I was at the Institute, were a happy, relaxed, and productive period for him. However in 1969 he had his first heart attack and from then on his heart condition was a cause of concern. In 1982 it came to a serious crisis and he understood from medical information that the prospects of recovery were dim. Here his family provided invaluable support and for the sake of his safety his daughters, Premi and Dini, accompanied him on his long and frequent walks.

It cheered him greatly to see again many old friends and colleagues at a conference at the Institute in October 1983. Always the perfect gentleman, his spirit seemed to have lifted from the gloom of previous weeks. His warm handshake and radiant smile gave some of us a ray of hope; more likely it meant that he himself had come to terms with his bitter fate. On Sunday 16 October he and Lily had many of the participants at their home. After the guests had departed, he went for his customary walk, but collapsed on the way, was found by passers-by and pronounced dead at the hospital. His ashes were spread in Princeton and immersed in the Ganges near Allahabad.

He lives on in the cherished memories of his friends and in his *Collected Works* which are a must for any serious student of representation theory. As I mentioned in the beginning, they have proved to be a fertile ground for mathematical progress in many different directions of which I have mentioned a very significant one 'Harmonic analysis on semisimple symmetric spaces'. Thus it is certain that Harish-Chandra's works will live for a long time as a brilliant chapter in the history of mathematics.

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