On the work of P-L. Lions

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During the International Congress of Mathematicians held at Zurich in August 1994, Lions was awarded Fields Medal for his significant contributions in the basic understanding of ordinary and partial differential equations, calculus of variations and other related areas. He has introduced several new ideas including the notions of renormalized solutions, viscosity solutions and the principle of concentration–compactness. The object of this article is to introduce certain physical models and show how fundamental issues associated with them can be settled by means of the above concepts.

THE International Congress of Mathematicians (ICM) takes place once in four years. The last one was held at Zurich, Switzerland, in August 1994. As usual, there were four recipients of Fields Medal, which is the equivalent of Nobel Prize in Mathematics. They are J. Bourgain, P-L. Lions, J-C. Yaccoz and E. Zelmanov. Apart from these, A. Wigderson received the Navanlinna Prize. These awards are usually given in recognition of the existing works as well as the promise of future achievements. These medalists were mostly honoured for a body of work rather than a single result except perhaps Zelmanov whose contribution was the solution of the Restricted Burnside Problem in Algebra. The other three awardees of Fields Medal are analysts. Yaccoz, one of the leading theorists in dynamical systems, is working in Université de Paris – Sud, Orsay, France. Bourgain (IHES, France) received the medal for his outstanding contributions to several areas of analysis including the geometry of Banach spaces, Harmonic analysis, Ergodic theory and nonlinear evolution equations.

In the above list of awardees, P-L. Lions stands out because he is probably the first medalist who has deep interest in a wide variety of applications. In a sense, this award to Lions may be an indication of the manner in which applications in many areas are now beginning to stimulate basic research in mathematics.

Before describing some aspects of his work, let us look into his professional life. Born in 1956 at Grasse, France, Pierre-Louis Lions was a product of Ecole Normale Supérieure and he obtained his Ph D from Université Pierre et Marie Curie, Paris in 1979. Ever since 1981, he is a professor attached to the Université Paris – Dauphine. To his credit, he has numerous awards and distinctions including membership of Academie des Sciences, Paris. He has been a regular speaker in the past few ICMs and other important conferences all over the world.

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Lions is an applied mathematician working in various aspects of differential equations (ordinary, partial, integro-differential, linear, nonlinear, etc.). He has made significant contributions in the basic understanding of ODE, PDE, calculus of variations and other related areas. Motivated from applications, he has tackled several mathematical questions, regarding various physical models which include Boltzmann equation, Vlasov–Poisson system, Hamilton–Jacobi equation, Hartree–Fock equation, etc. His existence and uniqueness results give a new insight into these equations which is essential in any numerical computations with them. Recently in ref. 12, he has obtained global weak solutions to compressible Navier–Stokes equations governing the flows of compressible viscous fluids, a very satisfying result after Leray's classical theorems on incompressible flows proved in 1930s. No doubt, the difficulties present in such models are due to the nonlinearities. A crucial concept which plays essential role in this work is that of renormalized solutions which is discussed below in Part A in the context of Boltzmann equation.

It will be a futile effort to try to survey all of Lions' works in one article and so a selection of the material has to be made. Apart from giving the references to the original articles, the purpose of this write-up is to give a glimpse of the work of Lions to mature scientists who are not necessarily professionals in Applied Mathematics. Having said this, it would be impossible for me to make everything accessible to non-mathematicians. After a certain point, mathematical technicalities enter the picture. Nevertheless, efforts are made to offer heuristic explanation whenever possible, stress the difficulties, highlight the ideas of Lions to overcome them and to set aside the technicalities as much as possible.

Part A: Kinetic equations and transport theory

Kinetic equations are fundamental mathematical models of Statistical Physics describing the dynamics of molecules in a rarefield gas. They provide a picture which is intermediate to Classical Mechanics and Continuum Mechanics. The state of the system is given by a non-negative function $f=f(t,x,v)$ which represents the density of molecules at time $t \in R^+$, at the point $x \in R^3$ and with velocity $v \in R^3$. Basic models which govern the evolution of f are of the following form:

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f+F\cdot\nabla_v f=Q(f,f).$$

Such models are dictated by the following reasonable picture: Since the gas is rarefield, there will be some time between successive collisions of molecules. During this time, the molecule will travel a positive distance called mean free path and its motion is governed by Newton's Law:

$$\frac{dx}{dt}=v,\ \frac{dv}{dt}=F.$$

Here $F=F(t,x,v)$ is the force acting on the molecule. The right hand side Q is called collision term which takes into account collision effects on f.

There are a lot of assumptions (implicit/explicit) in the construction of these models: molecular chaos, binary collisions, closeness to equilibrium, etc. Thus it is not clear even from the physical point of view that there is a global solution to them when the Initial Condition (IC) is far from equilibrium. Earlier there were some results covering some special cases: IC being very small, solutions for small time, homogeneous case (i.e. f independent of x), etc. No general result was available in the literature. That is why the general existence results of DiPerna and Lions[1] validating the above hypotheses came as a surprise. It is our aim here to see the new elements introduced by them in the general understanding of kinetic equations that led them to their success.

Boltzmann equation. Here $F=0$. The collision term Q is fairly complicated and it was constructed by Boltzmann using his famous molecular chaos hypothesis.

$$Q(f,f)(t,x,v)=Q_+(f,f)(t,x,v)-Q_-(f,f)(t,x,v),$$

$$Q_+=\int_{R^3}dv_*\int_{S^2}d\omega B(v-v_*,\omega)f'f'_*,$$

$$Q_-=\int_{R^3}dv_*\int_{S^2}d\omega B(v-v_*,\omega)ff_*,$$

$$f_*=f(t,x,v_*),\ f'=f(t,x,v'),$$

$$f'_*=f(t,x,v'_*),$$

$$v'=v-\langle v-v_*,\omega\rangle\omega,$$

$$v'_*=v_*+\langle v-v_*,\omega\rangle\omega.$$

Here $B=B(z,\omega)\geq 0$ is called the collision kernel and it is a function of $|z|$ and $\langle z,\omega\rangle$ only. Exact assumptions on B are not important for us here; it suffices to say that the approach of DiPerna and Lions allows kernels in a large class. Let us note the convolution structure in Q_- w.r.t the variable v:

$$Q_-(f,f)=f\cdot A*f,$$

$$A(z)=\int_{S^2}B(z,\omega)d\omega.$$

With all these notations, we are now in a position to write down the so-called Cauchy problem.

$$\left.\begin{array}{l}\text{Given } f_0=f_0(x,v)\geq 0,\qquad \text{find } f\geq 0 \text{ satisfying}\\ (\partial f/\partial t)+v\cdot\nabla_x f=Q(f,f)\ \text{ in } R_t^+\times R_x^3\times R_v^3\\ f(0,x,v)=f_0(x,v).\end{array}\right\}\quad (B)$$

1. *A general scheme for solving Cauchy problems and difficulties with Boltzmann equation*

The following general method has been found successful in the resolution of Cauchy problems for many models and DiPerna and Lions follow it to solve (B).

First step. (consistent approximation) Define a sequence of approximate solutions which can be easily constructed. In the case of (B), many such schemes are available. One such consistent approximation is as follows: approximate B by $\bar{B}\in C_0^\infty(R^3\times S^2)$ and Q by $\tilde{Q}$, where

$$\tilde{Q}(f,f)=\left(1+\delta\int f\,dv\right)^{-1}\bar{Q}(f,f),\qquad \delta>0,$$

where $\bar{Q}$ is the collision term corresponding to $\bar{B}$. Owing to the fact that $\tilde{Q}$ has linear growth w.r.t f (in contrast to Q which has quadratic growth), it is classical to establish the existence and the uniqueness of approximate solutions. The method is based on differentiating the approximate equation and thereby extracting bounds on the derivatives of its solution. (That is why B was smoothened out.) The task is by no means easy and is highly technical. Moreover, the crucial work of DiPerna and Lions does not lie here. Indeed this scheme of constructing approximating solution has been considered by earlier authors in the field. What was lacking is a proof that these approximate solutions converge to a suitable solution to (*B*). The contribution of DiPerna and Lions fills up this gap and provides the first global existence result for (*B*) via a constructive procedure.

To analyse this convergence, a sequence of *exact* solutions $\{f^n\}$ of (B) is considered in the sequel. It is to be reiterated that this is done only for simplicity and the treatment of approximating sequence will be parallel.

Second step. (stability) Extract maximum possible *a priori* estimates on $\{f^n\}$ independent of n. This depends a lot on the structure of the equation under consideration.

Third step. (convergence) By general results of Functional Analysis, it then follows that there exists a subsequence of $\{f^n\}$ which converges weakly to f (weak convergence simply signifies the convergence of the averages, e.g.: $\sin nx$ converges weakly to 0). It is a general belief that a consistent scheme which is stable is actually convergent. For reasons explained below, the truth of the above belief is not obvious in the case of nonlinear equations. That is why it is not easy to verify that f solves the problem (B).

2. A priori estimates

The basic idea is to find physical invariants of the form $\iint \psi f \, dx dv$ when f evolves according to (B) and $\psi = \psi(t, x, v)$ is a suitable function. Usual conservation laws of mass, momentum, energy, etc. are obtained in this way. It is important to realize that such invariants provide estimates on f in terms of IC. Indeed, exploiting the structure of Q, DiPerna and Lions derive the following estimate for all times t:

$$\iint (1 + |x|^2 + |v|^2) f \, dx dv \leq 2\iint (1 + |x|^2 + (1 + t^2)|v|^2) f_0 \, dx dv.$$

On the other hand, the technique of Boltzmann's H-theorem yields the inequality

$$\iint f |\log f| dx dv + \int_0^\infty \iint e(f) dx dv \, ds \leq c + \iint (|\log f_0| + 2|x|^2 + 2|v|^2) f_0 \, dx dv,$$

where $e(f)$ is the entropy flux density given by

$$e(f) = \frac{1}{4}\iint B(v - v_*, \omega)(f' f'_* - f f_*) \log\left(\frac{f' f'_*}{f f_*}\right) dv_* \, d\omega$$

and c is a numerical constant.

3. Extracting a convergent subsequence

In view of the above estimates, it is reasonable to impose the following restriction on IC:

$$\iint f_0 (1 + |x|^2 + |v|^2 + |\log f_0|) dx dv \leq C. \quad (1)$$

(Physically, this is a reasonable assumption. It says that the density, energy, entropy are finite initially. Earlier works assumed other smoothness properties on f_0 which are not physically meaningful.) Then the sequence of exact solutions $\{f^n\}$ will satisfy

$$\sup_{0<t<T} \iint (1 + |x|^2 + |v|^2 + |\log f^n|) f^n dx dv \leq C_T, \quad (2)$$

$$\int_0^\infty \iint e(f^n) dx dv ds \leq C. \quad (3)$$

In general, a bounded sequence $\{g^n\}$ in $L^1(\mathrm{R}^d)$, (which means that $\int_{\mathrm{R}^d} |g^n| dy \leq C$) can exhibit the following behaviours.

Case (i). The mass can escape to infinity. By this, it is meant a situation where

$$g^n(y) = g(y - y_n) \text{ with } y_n \to \infty.$$

and the support of g is bounded.

Case (ii). The mass can concentrate on null-sets. A trivial example is provided by Friedrichs' mollifiers: $g^n(y) = n^d \rho(ny)$ with $\rho \geq 0$ smooth, $\int \rho = 1$. It is known that g^n approaches Dirac measure δ_0 concentrated at the origin.

Case (i) is prevented here by the presence of weights $|x|^2$ and $|v|^2$ in equation (2). Case (ii) is ruled out because of the estimate on $f^n |\log f^n|$ implied by equation (2). Thus we conclude that (Dunford–Pettis criterion) there exists a subsequence (still denoted by n) such that

$$f^n \rightharpoonup f \text{ in } L^1((0, T) \times \mathrm{R}^3_x \times \mathrm{R}^3_v) \text{ weak.} \quad (4)$$

(Recall that this simply means that the averages of f^n converge to those of f.)

4. Difficulties

We have carried out Second step and Third step of the general scheme in §2 and §3 above. It is expected that the limit f obtained in equation (4) is a solution to (B). However, there are serious difficulties in proving this. We list them now.

(d1) Using equation (4), it is easy to pass to the limit in the left side of the Boltzmann equation in (B) because it is linear w.r.t f. On the other hand, it is not clear how to pass to the limit in the collision term because it is nonlinear. This is because a weakly convergent sequence (as in equation (4)) can sustain oscillations on finer and finer scales (e.g: $\sin nx \rightharpoonup 0$ weakly). In such a case, nonlinear functionals J can behave badly in the sense that $J(f^n)$ need not converge to $J(f)$ weakly (e.g: $\sin^2 nx \rightharpoonup 1/2$ weakly). What we are saying is that nonlinearities and averages do not commute which is a well-known fact. This difficulty is present in any nonlinear problem and in particular in (B).

(d2) If elliptic operators are involved then these oscillations are usually killed, i.e. we can obtain estimates on the derivatives of f^n. This phenomenon can be illuminated by considering Burgers equation which contains the heat operator:

$$\frac{\partial u^n}{\partial t} + u^n \frac{\partial u^n}{\partial x} = \frac{\partial^2 u^n}{\partial x^2}.$$

Multiplication by u^n yields bounds for $(\partial u^n/\partial x)$, $(\partial u^n/\partial t)$ and u^n. It is then common to use Rellich's Lemma (or more precisely, Lions' Lemma) which implies strong convergence of a subsequence of u^n (i.e. absence of oscillations in u^n). Using this, it is easily concluded that

$$u^n \frac{\partial u^n}{\partial x} \rightharpoonup u \frac{\partial u}{\partial x} \text{ weakly}$$

provided that

$$u^n \rightharpoonup u \text{ weakly.}$$

The conclusion is that the difficulty mentioned in (d1) disappears in the Burgers equation. Unfortunately in the present case, we have the following transport operator which constitutes the linear part of Boltzmann equation:

$$T \equiv \frac{\partial}{\partial t} + v \cdot \nabla_x .$$

This is hyperbolic and it does not kill oscillations. On the contrary it 'propagates' oscillations along real characteristics which it possesses. A simple example is provided by the equation $u_t + cu_x = 0$ with IC $u(x, 0) = g(x)$. Solution is explicitly given by $u(x, t) = g(x - ct)$ which shows clearly how initial values are propagated. In this example, solution is no smoother than IC, i.e. there is absolutely no smoothing effect at all. Let us close this discussion by mentioning that it is an open problem to derive estimates on derivatives of solutions of (B).

(d3) Now we see another difficulty which is by far the most serious one. As noted previously, Q contains essentially the convolution product w.r.t v and point-wise product w.r.t x, t. By Young's Inequality, convolution product of two integrable functions make sense as an integrable function whereas their point-wise product does not make sense because the averages of the product may not exist. Thus, for f satisfying bounds (2), it is not clear how to define the quantity $Q(f, f)$ in any reasonable way.

We know how to make distributions out of non-integrable functions in some simple situations. But the situation at hand is quite complicated. Of course, this difficulty can be overcome via Hölder Inequality if one can deduce L^p bounds ($p > 1$) on f. However, the derivation of such bounds for (B) is an open problem.

5. *Main result*

Because of the above difficulties, no genuine progress could be made for several decades. This is why, it came as a pleasant surprise when DiPerna and Lions[1,2] came up with a proof of the following result by introducing the notion of renormalized solutions:

Theorem 1. *Let f_0 satisfy equation* (1). *Then the weak limit f obtained in equation* (4) *satisfies* (*B*) *in the renormalized sense. Moreover f satisfies the entropy inequality*:

$$\iint f \log f \,\mathrm{d}x\mathrm{d}v - \iint f_0 \log f_0 \,\mathrm{d}x\mathrm{d}v + \iint e(f)\mathrm{d}x\mathrm{d}v \le 0 \quad \text{for all } t > 0.$$

It is classically known that a solution of Boltzmann equation in the usual sense formally satisfies entropy equality. This is the content of Boltzmann's H-theorem. Since f satisfies (B) in the renormalized sense, the question arises whether f has the same property. It turns out the solution constructed by DiPerna and Lions satisfies entropy inequality, thus obeying the all important Second Law of Thermodynamics.

In the sequel, we indicate how DiPerna and Lions overcome the above mentioned difficulties in proving Theorem 1.

6. *Renormalized solutions*

A fundamental problem in Applied Mathematics and Numerical Analysis is to characterize the weak limit of a sequence of approximate/exact solutions of a nonlinear system. For reasons cited above, this is a non-trivial problem. Turbulence modelling falls in this category.

Let us cite the success story of Hopf equation:

$$u_t + H(u)_x = 0, \quad x \in \mathrm{R}, \; t > 0, \; H(u) = (u^2/2),$$
$$u(x, 0) = u_0(x), \quad x \in \mathrm{R}.$$

Approximate solutions are defined by the introduction of artificial viscosity:

$$\left.\begin{aligned} u_t^\varepsilon + H(u^\varepsilon)_x &= \varepsilon u_{xx}^\varepsilon, \\ u^\varepsilon(x,0) &= u_0(x). \end{aligned}\right\} \tag{5}$$

The intrinsic characterization of the weak limit u of u^ε is as follows:

$$\left.\begin{aligned} &\eta(u)_t + q(u)_x \le 0 \;\; \forall \eta \text{ convex,} \\ &\quad \text{where } q' = \eta' H', \text{ prime denoting derivative w.r.t } u. \end{aligned}\right\} \tag{6}$$

This is somewhat surprising because we pass from an equation (5) to a set of inequalities (6). This can be easily deduced by multiplying equation (5) by $\eta'(u^\varepsilon)$ which yields

$$\frac{\partial}{\partial t}\eta(u^\varepsilon) + \frac{\partial}{\partial x} q(u^\varepsilon) = \varepsilon \eta(u^\varepsilon)_{xx} - \varepsilon \eta''(u^\varepsilon)(u_x^\varepsilon)^2 \le \varepsilon \eta(u^\varepsilon)_{xx}.$$

Passing to the limit as $\varepsilon \to 0$, we arrive formally at equation (6). Equation (6) is known as entropy inequality which is imposed on solutions of Hopf equation. It is known that such solutions are unique. In this process of characterization of the limit of u^ε, the main idea is the consideration of nonlinear transformation $\eta(u)$ of solutions.

Imitating the same idea, DiPerna and Lions introduce:

Definition. $f \in L^1_{loc}$ (i.e. f is locally integrable) is a renormalized solution of (B) if $Q_\pm(f,f)(1+f)^{-1} \in L^1_{loc}$ and the following equation is satisfied in the sense of distributions (i.e. in the sense of averages):

$$\left.\begin{aligned}\frac{\partial g}{\partial t}+v\cdot\nabla_x g = Q(f,f)(1+f)^{-1},\\ \text{with } g=\log(1+f).\end{aligned}\right\} \qquad \text{(RB)}$$

The choice of the nonlinear transformation $\beta(f) \equiv \log(1+f)$ may be justified because (i) it helps to reduce the growth of nonlinearities, (ii) log plays an important role in (B) since the time of Boltzmann. In general, the notions of renormalized and distributional solutions do not coincide because we are multiplying by zero at points where $f \to \infty$. They do if $Q_\pm \in L^1_{loc}$. Formally, (RB) is obtained by dividing (B) by $(1+f)$. So the above definition might look trivial. But it must be remembered that equation in (B) does not make sense in general whereas it is proved that (RB) does. The significance of this new concept is that the weak limit of nice solutions is a renormalized solution that need not be a distributional solution of (B). This point was completely missed by the earlier researchers in the field. Renormalization is the technique used by DiPerna and Lions to overcome the difficulty (d3).

7. *Compactness of velocity averages*

Here we see a remedy suggested by Lions *et al.*[5] to overcome the difficulty (d2) listed above. If there is a smoothing effect, (e.g. estimation on the derivatives of f) then weak convergence becomes strong via, for instance, *Rellich's Lemma.* This can then be used to handle nonlinearities as seen above. However in the case of transport operator T, there is smoothing only in non-characteristic directions. In the characteristic direction v; singularities propagate. The discovery in ref. 5 is that velocity averaging compensates for the lack of regularization in that direction. In some loose terms, one may state that the average of a family of hyperbolic operators is an elliptic operator. More precisely.

Theorem 2. *Let* $g \in L^2(\mathrm{R}_t \times \mathrm{R}^3_x \times \mathrm{R}^3_v)$ *be such that* $\operatorname{supp} g = K$ *is compact and* $Tg = G \in L^2$. *Then* $\int g\,\mathrm{d}v \in H^{1/2}(\mathrm{R}_t \times \mathrm{R}^3_x)$.

Let us explain the significance of the above result without defining precisely the symbols used. It says that a solution g of the transport equation $Tg = G$ gains 1/2-derivative provided we average out w.r.t v. Indeed, it can be taken as a quantitative statement of the feeling that 'averages smoothen' out things. Such a result is to be compared with a classical one involving Laplacian. In fact if $\Delta g = G \in L^2$ then by Fourier transform $\hat{G}(\xi) = |\xi|^2\,\hat{g}(\xi)$ and so by Parseval's Identity we see that $(\partial g/\partial x_i)$, $(\partial^2 g/\partial x_i \partial x_j)$ are in L^2. Thus we have a gain of 2-derivatives here.

Proof of Theorem 2. Taking Fourier transform w.r.t x, the main point is to estimate

$$\int_{\mathrm{R}^3} |\xi| \left| \int_{K_\alpha} \hat{g}(\xi,v)\mathrm{d}v \right|^2 \mathrm{d}\xi,$$

where $K_\alpha = \operatorname{supp} \hat{g}(\xi,\cdot) \cap \{|\xi.v| \le \alpha\}$. By Cauchy–Schwarz inequality, this is dominated by

$$\int_{\mathrm{R}^3}\int_{\mathrm{R}^3} |\xi| \operatorname{meas}(K_\alpha) |\hat{g}(\xi,v)|^2 \,\mathrm{d}\xi \mathrm{d}v.$$

The proof is over if we remark that

$$\operatorname{meas}(K_\alpha) \le C_\alpha\,(\alpha/|\xi|).$$

8. *Passage to the limit*

In this part, which is very technical and tedious, DiPerna and Lions show how to overcome the difficulty (d1). The main observation is that $Q(f,f)$ involves velocity averages and the result of §7 shows that there is then a smoothing effect. We have already illustrated the passage to the limit in nonlinear terms in such circumstances. We end here our discussion on Boltzmann equation.

9. *Ordinary differential equations*

In this paragraph, we see another beautiful piece of work of DiPerna and Lions where they tackle a problem which appears in many practical situations. Given a vector field $B(x)$, $x \in \mathrm{R}^N$, we are interested in the flow $X(t,x)$ defined by it: $(\mathrm{d}X/\mathrm{d}t) = B(X)$, $X(0,x) = x$. The first result which comes to our mind is the classical Cauchy–Lipschitz Theorem which assumes B is Lipschitz and proves that for a given $x \in \mathrm{R}^N$, there is a unique trajectory $t \mapsto X(t,x)$ passing through x. This is the pointwise description of the flow. The measure transported by this flow satisfies

$$\mathrm{e}^{-c_0|t|}\lambda \le \lambda \circ X_t \le \mathrm{e}^{c_0|t|}\lambda \quad \forall t \in \mathrm{R}, \qquad (7)$$

where λ is the Lebesgue measure and c_0 is the maximum value of $|\operatorname{div} B|$. In particular, equation (7) shows that Lebesgue measure is preserved, if $\operatorname{div} B = 0$. Thus equation (7) may be considered as a generalization of Liouville's Theorem. Furthermore, we have also the stability of individual trajectories in terms of IC:

$$|X(t, x_1) - X(t, x_2)| \leq e^{c_1|t|}|x_1 - x_2|, \quad (8)$$

where c_1 is the Lipschitz constant of B.

In many practical situations, B is not Lipschitz. For instance, in incompressible fluid flows governed by Navier–Stokes equations, the velocity field satisfies $\text{div}\, u = 0$ and ∇u is square integrable, i.e. $\nabla u \in L^2$ which signifies that the energy of the flow is finite. On the other hand, we cannot assert that u is Lipschitz. As a consequence, Cauchy–Lipschitz Theorem is inapplicable and so a Langrangian description of the flow is not available. This has been one of the major handicaps in advancing our knowledge on the qualitative behaviour of such flows. The work of DiPerna and Lions[4] removes this obstacle and paves the way for further developments.

The inequality (8) shows the instability of individual trajectories if B is not Lipschitz. The idea of DiPerna and Lions to overcome this difficulty is to reject the point-wise picture of the flow and pass to almost-everywhere flow. Initial condition is not fixed. In fact, a null set of IC causing instabilities is thrown away. With this in mind, DiPerna and Lions introduce the following definition. In order to eliminate the problems at infinity and concentrate more on the lack of regularity of B, we consider bounded smooth region $\Omega \subset \mathbb{R}^N$ which is invariant, i.e. $B(x)\nu(x) = 0$, where $\nu(x)$ is the unit exterior normal at $x \in \partial\Omega$. This signifies that fluid is in a container which it does not leave.

Definition. X is an almost everywhere flow associated to B if (a) $X \in C^0(\mathbb{R}; L^1(\Omega))$, (b) $X(t, x) \in \overline{\Omega}$, $\forall t \in \mathbb{R}$, $x \in \Omega$ a.e., (c) $(\partial X/\partial t) = B(X)$ in the sense of distributions. (d) $X(t + s, x) = X(t, X(s, x))$ $\forall t, s \in \mathbb{R}$, $x \in \Omega$ a.e. (e) (7) holds.

Theorem 3. *Let* $B \in (L^1(\Omega))^N$, $\nabla B \in (L^1(\Omega))^{N \times N}$ *and div B be bounded. Then there is a unique a.e flow X such that* (7) *holds. Further the flow X is stable w.r.t perturbations on B.*

The measure transported by the flow becomes important in this analysis. Because this measure satisfies equation (7), it is intuitively clear that it is enough to require for stability reasons that $\text{div}\, B$ is bounded. The requirement that ∇B is integrable in the above Theorem comes up as a consistency condition in the course of analysis which cannot be explained in simple terms. Let us simply remark that these conditions are considerably weaker than Lipschitz condition and they are satisfied by incompressible fluid flows. Existence of unique Hamiltonian flows is something basic in Classical Mechanics and taken for granted by physicists. It is to be pointed out that this was not guaranteed in general before the work of DiPerna and Lions. In the past, there have been some attempts to define flows for singular vector fields (Piano's solutions, Fillipov solutions, Glimm-Lax solutions). All of them are 'local' whereas the point of view of DiPerna and Lions is 'global' in the sense that it is whole flow which is now defined and not individual trajectories. The secret of their success is discussed in the next section. We close this section by mentioning that they have obtained several interesting applications of their theory in Fluid Mechanics. We will see below in §11 one application in Kinetic Theory.

10. Linear transport equations

It is usual to deduce qualitative properties of PDEs by reducing them to ODEs (e.g. method of characteristics). However in the present case, DiPerna and Lions follow a rather unconventional method of passing from Lagrangian picture (ODE) to Eulerian picture (PDE). Such a study is rendered possible thanks to the notion of renormalized solutions.

The Euler equation corresponding to the flow $X(t, x)$ is obtained by writing down the PDE satisfied by $u(x, t) = u_0(X(t, x))$, where u_0 is a given initial condition (IC). Assuming $\text{div}\, B = 0$ (for simplicity), we see that u satisfies

$$\left.\begin{array}{l}(\partial u / \partial t) = div\,(B(x)u) \text{ in the sense of distributions,} \\ u(x, 0) = u_0(x) \;\; x \in \Omega.\end{array}\right\} \quad (9)$$

This is a linear PDE in contrast to the fact that the initial system of ODEs was nonlinear. However, the coefficient $B(x)$ is not smooth. The idea is to solve equation (9) for a large class of u_0 which will enable us to prove Theorem 3. However there is a difficulty in solving equation (9). *If* $u_0 \in L^1$ *then* $u \in L^1$ *and so the product Bu is not* defined as a distribution. This situation is familiar to us and a remedy has already been suggested in §6. The idea is to consider nice nonlinear transformation $\beta(u)$ and introduce

Definition. $u \in L^\infty$ $(0, T; L^1(\Omega))$ is a renormalized solution of equation (9) if $\beta(u)$ is a distributional solution of equation (9) with IC $\beta(u_0)$ $\forall\, \beta \in C_0^1(\mathbb{R})$.

The two notions (distributional and renormalized solutions) are, in general, different but coincide if $u \in L^\infty$ $(0, T; L^\infty(\Omega))$. To prove the existence, DiPerna and Lions approximate B and u_0 by B^ε and u_0^ε obtained by convolution with Friedrichs' mollifiers. It is then classical to solve equation (9) with B^ε and u_0^ε instead of B and u_0. Let the solution be denoted as u^ε. The big task is to show that u^ε converges weakly and to characterize the weak limit. To carry out this, they follow the scheme described in §1. Since the method was already explained in some detail in the case of Boltzmann equation, we do not intend to repeat it here. The main point is that such a scheme converges to a renormalized solution of equation (9) and not to a distributional solution. The new progress that can be made now and that was not possible in (B) is a proof of uniqueness for equation (9). The proof is a technical marvel which combines all possible tricks (due to Stampacchia, Moser, Kruzkov, Gronwall, Holmgren ...). Their end result is

Theorem 4. *There is a unique renormalized solution u for equation (9).*

11. Vlasov–Poisson system

This system can be presented in the following form:

$$\left.\begin{array}{l} \text{Given } f_0 = f_0(x,v) \geq 0, \quad \text{find } f = f(t,x,v) \geq 0 \\ \text{satisfying } f(0,x,v) = f_0(x,v) \text{ and} \\ (\partial f/\partial t) + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \text{ in } \mathrm{R}_t^+ \times \mathrm{R}_x^3 \times \mathrm{R}_v^3 \\ F = F(t,x) = -\nabla_x \Phi(t,x) \\ -\Delta\Phi = \rho \text{ in } \mathrm{R}_t^+ \times \mathrm{R}_x^3, \\ \rho(t,x) = \int_{\mathrm{R}^3} f(t,x,v)\mathrm{d}v. \end{array}\right\} \quad \text{(VP)}$$

It is proved by DiPerna and Lions[6] that there is a renormalized solution $f \geq 0$ to (VP) under suitable hypothesis on f_0. It is tempting to compare (VP) with (B). We are going to discuss the main difficulty in (VP) which was not present in (B). The above system is simpler than (B) in that collision term is absent and more complicated in that the external force F is nonzero and is coupled to f in a complicated nonlocal way giving rise to nonlinearities. A quick look at the system tells that we must define the flow in (x, v) space corresponding to the vector field

$$B(t, x, v) = (v, -\nabla_x \Phi(t, x)).$$

The principal problem is that this vector field is not smooth enough and we cannot apply Cauchy–Lipschitz Theorem. On the other hand, the theory of DiPerna and Lions presented in §9 applies here. First of all, $\mathrm{div}_{x,v} B = 0$. Further the following can be said regarding the regularity of Φ through a chain of sharp statements: The available estimates on f solution of (VP) are

$|v|^2 f \in L^1_{x,v}$ (finite energy property),

$f \log^+ f \in L^1_{x,v}$ (finite entropy property).

These imply that $\rho \log^+ \rho \in L^1_{\mathrm{loc}}$. For non-negative functions ρ, this condition is equivalent to saying that ρ is an element of the Hardy class $\mathcal{H}^1_{\mathrm{loc}}$. Thus we see how entropy condition and Hardy space are linked.

The question is how to deduce the regularity property of Φ which is connected to ρ via the equation $-\Delta\Phi = \rho$. We have already seen in §7 that we gain two derivatives in L^2 sense for Φ. Unfortunately, such a result is not true in L^1 sense. More exactly, if $\rho \in L^1_{\mathrm{loc}}$ then we cannot assert $(\partial\Phi/\partial x_i)$, $(\partial^2\Phi/\partial x_i\, \partial x_j) \in L^1_{\mathrm{loc}}$. Since Hardy space regularity is better than mere integrability and since $\rho \in \mathcal{H}^1_{\mathrm{loc}}$ the above regularity for Φ is indeed true and this can be deduced from the theory of Singular Integrals. We conclude therefore that $\nabla B \in L^1_{\mathrm{loc}}$. Thus the crucial hypothesis of Theorem 3 is satisfied and so we can associate a unique flow to (VP) which solves the problem.

Among several applications given by DiPerna and Lions, we have chosen (VP) system to show the sharpness of the results involved so that the reader can appreciate the hypotheses and the conclusions of Theorem 3 in a better way.

Part B: Theory of compensated compactness and harmonic analysis

A basic problem with nonlinear models coming from various fields is that nonlinear terms appearing in the equation are not continuous w.r.t the weak topology in the space defined by the physical estimates available for the model. In other words, averaging and nonlinearities do not go hand in hand. The reason is that a weakly convergent sequence can sustain oscillations which are magnified by unstable nonlinearities. This was illustrated by a simple example in Part A.

This was a great stumbling block in the understanding of many models from mechanics, for instance. A new era started with the work of J. Ball around 1977 when he discovered physically meaningful nonlinear functionals which are continuous w.r.t weak topology. The purpose of this part is to describe this situation in detail in one example and point out the significant contribution of Lions in this domain.

The situation we have in mind concerns nonlinear elasticity. Under the action of a force field, the elastic body undergoes deformations and its state is described by a displacement field $u = (u_1, u_2, u_3)$ which minimizes suitable energy functional. The admissible fields satisfy $\nabla u \in L^3(\mathrm{R}^3)$, i.e. $\int|\nabla u|^3\mathrm{d}x$ is finite which simply signifies that the stored energy of the material is finite. If the material is incompressible then we have in addition that $\det(\nabla u) = 1$. In numerical computations, one is often led to ask: is the limit of a sequence of incompressible materials incompressible? The corresponding mathematical question can be posed as follows: consider a sequence of fields $\{u_n\}$ with bounded energy, i.e. $\int|\nabla u_n|^3\mathrm{d}x \leq c$ and $\det(\nabla u_n) = 1\ \forall n$. As usual it follows, from general results of Functional Analysis that for a subsequence $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^3(\mathrm{R}^3)$. Can one say that $\det(\nabla u) = 1$? (or) $\det(\nabla u_n) \rightharpoonup \det(\nabla u)$ in the sense of averages? Since nonlinear functionals, in general, behave badly w.r.t weak convergence, it was a great achievement when J. Ball answered the above question affirmatively.

Trying to generalize this example, Murat and Tartar came up with a theory (bearing the name of this part) in which a fine Fourier analysis of the oscillations and the functional is done to explain why certain nonlinear functionals behave nicely, i.e. they are weakly continuous.

In spite of its existence, the above result had remained somewhat a mystery because regularity was always the source of good behaviour and there is no mention of it in the above theory. Recalling that $\det(\nabla u)$ is a linear combination of terms of the form $(\partial u_i/\partial x_l)$ $(\partial u_j/\partial x_m)$ $(\partial u_k/\partial x_n)$, *a priori* we can only assert that $\det(\nabla u)$ is integrable, via Hölder inequality. That is why, people were pleasantly surprised when Lions *et al.*[11] brought Hardy space $\mathcal{H}^1$ into the picture and proved

Theorem 5. *For* $u \in W^{1,3}(\mathbb{R}^3)$, $\det(\nabla u) \in \mathcal{H}^1(\mathbb{R}^3)$.

We cannot possibly go into the definition of Hardy spaces here. In the last section, we have already seen a characterization of positive elements ρ of $\mathcal{H}^1$, namely that $\rho \log^+ \rho$ is integrable. From this, it is obvious that saying some function belongs to the Hardy space $\mathcal{H}^1$ is much more than saying that it is integrable. The reason for this good regularity given by above theorem can be heuristically explained as follows: It is true that each term $(\partial u_i/\partial x_l)$ $(\partial u_j/\partial x_m)$ $(\partial u_k/\partial x_n)$ is merely integrable. However, $\det(\nabla u)$ is formed by a particular combination of these terms and there are nontrivial cancellations between them leading to the above regularity. In other words, there is a compensation between this functional form and the given information $\nabla u \in L^3(\mathbb{R}^3)$. Hence the name of the theory.

This result not only explained those of Ball, Murat and Tartar but also improved upon them. Furthermore, it initiated what is now known as Hardy space approach to nonlinear problems. We end this section by pointing out Theorem 5 is only one among a set of results due to Lions in this direction.

Part C: Theory of concentration – compactness in calculus of variations

In several problems in mechanics and other fields, the ground state of a system admits a variational characterization of the following form: Find $u \in K$ such that

$$I = E(u) = \min_{v \in K} E(v). \qquad (10)$$

Typically, $E(v)$ is the energy functional of the system, v is the displacement field and K is the set of constraints. An elementary example is provided by considering the free elastic vibrations of a membrane Ω fixed on its boundary. The aim is to find the dominant frequency of these vibrations. Such a situation is modelled by equation (10) with

$$\left.\begin{array}{l}\Omega \text{ open bounded in } \mathbb{R}^N \\ E(v) = \int_\Omega |\nabla v|^2\,dx, \\ K = \{v \in H_0^1(\Omega);\ \int_\Omega |v|^2\,dx = 1\} \\ H_0^1(\Omega) = \{v; \int_\Omega |v|^2\,dx, \int_\Omega |\nabla v|^2\,dx \text{ are finite}, v = 0 \text{ or } \partial\Omega\}.\end{array}\right\} \qquad (11)$$

The choice of K signifies that the displacements in K have finite energy.

How to solve equation (10)? In practice, one is interested not only in the existence of a solution to equation (10) but also in its stability. That is why, one is interested in the convergence of any minimizing sequence to equation (10) and this is essential for computations. With this goal in mind, the so-called direct method of calculus of variations proceeds as follows:

First step. Take any minimizing sequence $\{v_n\}$, i.e. $v_n \in K$ and $E(v_n) \to I$. The goal is to examine the convergence of $\{v_n\}$ towards a ground state.

Second step. Since $E(v_n)$ is bounded, we will be able to deduce energy estimates on $\{v_n\}$.

Third step. Using general results of functional analysis, we conclude that some subsequence of $\{v_n\}$ will converge to a limit u weakly, i.e. in the sense of averages.

Fourth step. Assuming nice continuity properties of E w.r.t weak topology, we deduce that $\liminf E(v_n) \geq E(u)$. In particular, this implies $E(u) \leq I$.

Fifth step. To complete the proof that u is a ground state, we must finally show that $u \in K$.

In practical cases, it is usually difficult to carry out fourth and fifth steps. We have already had a taste of difficulties in Part A. The above programme has been successfully carried out in the case of equation (11). To verify Step (5), one uses Rellich's Lemma which implies that

$$\int_\Omega |v_n|^2\,dx \to \int_\Omega |v|^2\,dx$$

provided that $v_n \to v$ in $H_0^1(\Omega)$ weak.

One can generalize example (11) by choosing

$$K = \left\{v \in H_0^1(\Omega); \int_\Omega |v|^p\,dx = 1\right\}. \qquad (12)$$

This time, Step (5) is carried out via Sobolev's Lemma which says that the nonlinear functional $\int_\Omega |v|^p\,dx$ behaves well w.r.t weak convergence in $H_0^1(\Omega)$ provided Ω is bounded and $1 \leq p < (2N/N-2)$. In this sense, the number $(2N/N-2)$ is a critical index. If $p = (2N/N-2)$ in equation (12) then the above arguments fail. Indeed, the conclusion of the Sobolev's Lemma is violated for the sequence

$$u^\varepsilon(x) = \varepsilon^{-\alpha}u(x/\varepsilon), \quad \alpha = (N-2/2)$$
$$\text{(mass concentration at 0).} \qquad (13)$$

Likewise, with $\Omega = \mathbb{R}^N$, a counter example for Rellich's Lemma is provided by

$$u^n(x) = u(x + x_n) \text{ with } x_n \to \infty \text{ (mass going to infinity).} \qquad (14)$$

Thus we see that non-compact groups of translations and dilations are potential sources of difficulties and we are not able to carry out Step (5) of the above programme.

Lions' works[9,10] are devoted to the study of such non-compact situations. Several researchers have worked in the field. Perhaps Lions is not the first one to discover the phenomena involved in such cases. His originality lies in the new viewpoint regarding such problems, generality of his arguments and the enormous number of applications developed by him and his coworkers. Below, let us see one example where his theory applies.

Slightly modifying our earlier example, we consider

$$\left.\begin{aligned} I_\lambda &= \min_{v\in K}\int_{\mathbb{R}^N}\{|\nabla v|^2 + a(x)|v|^2\}\mathrm{d}x,\\ K &= \left\{v\in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N}|v|^p\mathrm{d}x = \lambda\right\}.\end{aligned}\right\} \qquad (15)$$

Here we take $a(x)$ smooth such that, $a(x) \geq a_0 > 0$ and $\lim_{x\to\infty} a(x) = a^\infty$, $\lambda > 0$ is a parameter in the problem. The index p is taken to be subcritical: $2 < p <(2N/N-2)$, since the problem (15) is posed on $\mathbb{R}^N$ which is unbounded. Step (5) of the direct method does not carry through for reasons described above (see equation (14)). This is one of the fundamental difficulties which arises in many practical contexts of which equation (15) is an example.

The idea of Lions to overcome this difficulty is to define a comparison problem using the same group (of translations) which is the source of obstacles. Accordingly, the following problem at infinity is introduced:

$$I_\lambda^\infty = \min_{v\in K}\int_{\mathbb{R}^N}\{|\nabla v|^2 + a^\infty|v|^2\}\,\mathrm{d}x. \qquad (16)$$

The main result can now be stated.

Theorem 7. *If the strict inequalities*

$$I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty \quad 0 \leq \alpha < \lambda \qquad (17)$$

are satisfied then any minimizing sequence for equation (15) *will admit a subsequence which will converge in* $L^p(\mathbb{R}^N)$. (*In particular, the ground state exists for* (15).) *The converse is also true.*

The primary task in the resolution of equation (15) is to locate the place where the energy density is concentrated. After all, the drifting of the energy towards infinity is the root cause of all troubles (see equation (14)). The goal is to examine the circumstances under which the energy is concentrated in a bounded region of the space which will reduce us to the situation of Sobolev's Lemma and so we are done. The method derives its name from these considerations.

Thus one is led to consider the energy density $f_n \equiv |\nabla v_n(x)|^2 + |v_n(x)|^2$ associated to any minimizing sequence $\{v_n\}$. Note that $\int f_n \mathrm{d}x \leq c$. What can be said of the possible behaviour of $\{f_n\}$? Surprisingly, there are *only three types of them which are mutually exclusive.* We describe them in words.

Vanishing. Energy does not concentrate anywhere. This is physically uninteresting and so eliminated.

Tightness. Energy in f_n is concentrated at a point $x_n \in \mathbb{R}^N$.

Dichotomy. Energy in f_n is split into two parts f_n^1, f_n^2 which are concentrated in two regions of the space which drift away w.r.t each other.

Dichotomy can occur and it has to be eliminated if we expect good behaviour of the minimizing sequence. That is why strict inequalities (17) are imposed. Indeed assuming $\int_{\mathbb{R}^N}|v_n^1|^p\,\mathrm{d}x = \alpha$ and $\int_{\mathbb{R}^N}|v_n^2|^p = \lambda - \alpha$, we see intuitively that $I_\lambda = I_\alpha + I_{\lambda-\alpha}^\infty$ if dichotomy were to hold. Next, if $\{x_n\} \to \infty$ in the case of tightness then the situation is similar to Dichotomy with $v_n^1 \equiv 0$ or equivalently $\alpha = 0$. Thus we would have $I_\lambda = I_\lambda^\infty$ which is again ruled out by equation (17). Thus, the only case left out is Tightness where $\{x_n\}$ is bounded. In this case, we can apply Sobolev's Lemma and so we are through.

Finally, it remains to verify the strict inequalities (17) and that depends on the problem at hand. In the case of equations (15), by homothecy, one can check that $I_\lambda = \lambda^{(2/p)} I_1$ and so equation (17) holds for $0 < \alpha < \lambda$. If $\alpha = 0$, equation (17) is reduced to $I_\lambda < I_\lambda^\infty$. Whether this holds or not depends on the way $a(x)$ tends to a^∞ as $x \to \infty$. Since this is technical, we stop our discussions here.

Part D: Hamilton–Jacobi equation

The H–J equation is a scalar, first order PDE proposed originally as one of the formulations to describe the classical mechanics of particles. We consider its simplest form

$$\frac{\partial\phi}{\partial t} + H(\nabla_x\phi) = 0, \quad x \in \mathbb{R}^N, \quad t > 0. \qquad \text{(HJ)}$$

where $H = H(p)$ is a smooth Hamiltonian. To see the amazing number of ways, it props up in various contexts, the reader is advised to go through Lions *et al.*[8]. Its connection with PDE theory can be seen as follows: Consider a linear partial differential operator of the form $P(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$. The plane waves generated by it are given by $u(x) = e^{i\xi\cdot x}$ where $\xi \neq 0$ is a cotangent vector in $\mathbb{R}^N$ satisfying $P(\xi) \equiv \sum_{|\alpha|=m} a_\alpha \xi^\alpha = 0$. In order to obtain more general waves generated, we look for solutions of the form $u(x) = e^{i\varphi(x)}$ satisfying

$P(D)u = 0$. This leads to nonlinear first order PDE of the type (HJ) for the scalar phase function ϕ even though the original equation is linear. Thus, it is imperative to understand (HJ) if one wants to study the instabilities associated with $P(D)$.

Another way to reach (HJ) is to consider the classical scalar conservation law:

$$u_t + H(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{18}$$

If $\phi(x, t) = \int_{-\infty}^{x} u(y, t)\mathrm{d}y$ then ϕ satisfies (HJ). Thus equation (18) can be considered as a subclass of (HJ) via the relation $\phi_x = u$ in the case $N = 1$.

One of the first difficulties in the resolution of Cauchy problem for equation (18) or (HJ) is that the method of characteristics fails to produce a global solution $u \in C^0$ for equation (18) or $\phi \in C^1$ for (HJ); the characteristics are described by Hamiltonian system which in this case is given by

$$\mathrm{d}x/\mathrm{d}t = H'(u(x, t)), \text{ prime-denoting derivative.}$$

It is easily checked that $u(x(t), t) \equiv$ constant if u satisfies equation (18). Hence, characteristics are straight lines with speed $H'(u(x, t))$. If the initial condition is such that $H'(u_l) > H'(u_r)$ then the characteristics corresponding to u_l and u_r would meet and we will have two values for the solution at the point of intersection. The basic question is, therefore, whether one can make a choice among such multiple values so that we have a smooth solution of (HJ) or (18). Looking at the complicated wave patterns (starting from smooth phase) produced in experiments, one concludes that this is neither possible nor realistic.

The next best thing is to accommodate such singular solutions in (HJ) and (18). Since equation (18) is in the divergence form, one can talk of bounded solutions. Since derivatives occur inside nonlinearity in (HJ), it is not obvious how to define a solution concept in the class C^0. Even if we do this, it is not guaranteed that there is a unique solution. In fact, one can easily produce multiple continuous solutions for (HJ) all satisfying same IC. Nonuniqueness had been one of the fundamental difficulties with (HJ) and (18). These issues were settled by Lax, Oleinik, Kruzkov, etc. in the case of equation (18) and more recently this analysis has been extended by Lions *et al.*[7] in a better way to the case of (HJ).

The principal task is therefore to identify the physically relevant solutions. One of the parameters which is assumed to be zero in (HJ) and in equation (18) is the viscosity. If we introduce it, we get the following unfoldings of the equations (HJ), (18) respectively:

$$(\partial\phi^\varepsilon/\partial t) + H(\nabla\phi^\varepsilon) = \varepsilon\Delta_x\phi^\varepsilon, \tag{19}$$

$$(\partial u^\varepsilon/\partial t) + H(u^\varepsilon)_x = \varepsilon\Delta_x u^\varepsilon. \tag{20}$$

It is physically reasonable to say that the limits of ϕ^ε, u^ε, as $\varepsilon \to 0$ are the meaningful solutions of (HJ) and equation (18) respectively. How to characterize these limits in an intrinsic fashion?

As seen in Part A, letting $\varepsilon \to 0$ in equation (20) leads to the set of inequalities (6) which are to be imposed on solutions of equation (18). Physically, this implies that only compressive shocks are allowed to be present in a solution of equation (18). With this additional requirement, one can prove existence, uniqueness and stability of solutions for equation (18).

How to extend the above considerations to equation (19)? Following Lions *et al.*[7], we can formally argue as follows: Assume $\phi^\varepsilon \to \phi$ uniformly. Let ψ be a smooth test function. Consider a local maximum $(x^\varepsilon, t^\varepsilon)$ of $\phi^\varepsilon - \psi$. At this point, we have $\nabla_{x,t}\phi^\varepsilon(x^\varepsilon, t^\varepsilon) = \nabla_{x,t}\psi(x^\varepsilon, t^\varepsilon)$ and $D^2\phi^\varepsilon(x^\varepsilon, t^\varepsilon) \leq D^2\psi(x^\varepsilon, t^\varepsilon)$. In particular, $\Delta_x\phi^\varepsilon(x^\varepsilon, t^\varepsilon) \leq \Delta_x\psi(x^\varepsilon, t^\varepsilon)$. As $\varepsilon \to 0$, $(x^\varepsilon, t^\varepsilon)$ will move towards a local maximum (x_0, t_0) of $\phi - \psi$. Moreover at this point, we have

$$(\partial\psi/\partial t) + H(\nabla\psi) \leq 0. \tag{21}$$

This motivates the following

Definition. We say $\phi \in C^0$ is a viscosity sub-solution if equation (21) holds at every local maximum (x_0, t_0) of $\phi - \psi$. Likewise, $\phi \in C^0$ is viscosity super-solution if $(\partial\psi/\partial t) + H(\nabla\psi) \geq 0$ at every local minimum (x_0, t_0) of $\phi - \psi$. Viscosity solutions are both sub and super solutions.

First remark is that if $\phi \in C^1$ is a classical solution then it is a viscosity solution because we can take $\psi = \phi$. The above definition completely avoids reference to the derivatives of ϕ. In distribution theory, we are used to shifting derivatives to test functions via integration by parts. Here using maximum principle, we are placing the derivatives on the test function inside the nonlinearities. Thus maximum principle is something in-built into viscosity solutions. It is therefore clear that this concept will be useful in systems which possess order-preserving property. The main properties enjoyed by viscosity solutions which are not shared by other solutions are listed in the following:

Theorem 8. We take the initial condition $\phi(x, 0) = \phi_0(x)$, $x \in \mathbb{R}^N$ and consider the Cauchy problem for (HJ). Under suitable growth conditions on H, we have

(i) *Existence of viscosity solutions ϕ in the class of Lipschitz functions.*
(ii) *Uniqueness of viscosity solutions in the class of bounded continuous functions.*
(iii) *Viscosity solutions are stable w.r.t uniform convergence on compact sets without involving derivatives.*
(iv) (*Comparison*) *If ϕ, ψ are viscosity solutions of (HJ) with IC ϕ_0 and ψ_0 respectively with $\phi_0 \leq \psi_0$ then $\phi \leq \psi$.*

The proof of uniqueness, is, without doubt, the significant part of the above result. Kruzkov proved the same for equation (18) among solutions of bounded variation assuming entropy condition. His proof is quite complicated and ingenious because one has to play with inequalities (6). Assertion in Theorem 8 (ii) is much stronger. However, the proof given in ref. 7 is surprisingly not very complicated.

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Some publications of P-L. Lions

On kinetic equations and transport theory

1. (with DiPerna, R. J.) On the Cauchy problem for Boltzmann equations. Global existence and weak stability, *Ann. Math.*, 1980, **130**, 321–366.
2. (with DiPerna, R. J.) Global solutions of Boltzmann equations and the entropy inequality, *Arch. Rat. Mech. Anal.*, 1991, **114**, 47–55.
3. (with DiPerna, R. J.) Global weak solutions of Vlasov–Maxwell systems, *Comm. Pure Appl. Math.*, 1989, **42**, 729–757.
4. (with DiPerna, R. J.) Ordinary Differential Equations, Sobolev spaces and transport theory, *Invent. Math.*, 1989, **98**, 511–547.
5. (with Golse, F., Perthame, B. and Sentis, R.) Regularity of the moments of the solutions of a transport equation, *J. Funct. Anal.*, 1988, **76**, 110–125.
6. (with DiPerna, R. J.) Solutions globales d'équations du type Vlasov–Poisson, *C. R. Acad. Sci., Paris*, 1988, t307, Série I, 655–658.

On viscosity solutions

7. (with Crandall, M. G.) Viscosity solutions of Hamilton–Jacobi equations, *Trans. Am. Math. Soc.*, 1983, **277**, 1–42.
8. (with Crandall, M. G. and Ishii, H.) User's guide to viscosity solutions of second order partial differential equations, *Bull. Am. Math. Soc.*, 1992, **27**, 1–67.

On the theory of concentration – compactness

9. The concentration–compactness principle in the calculus of variations, The locally compact case:

   Part 1. *Ann. I.H.P., Anal. Nonlin.*, 1984, **1**, 109–145.
   Part 2. *Ann. I.H.P., Anal. Nonlin.*, 1984, **1**, 223–283.
10. The concentration–compactness principle in the calculus of variations, The limit case:

   Part 1. *Riv. Mat. Ibero – americana*, 1985, **1**, 145–201.
   Part 2. *Riv. Mat. Ibero – americana*, 1985, **1**, 45–121.

On the theory of compensated compactness

11. (with Coifman, R., Meyer, Y. and Semmes, S.) Compacité par compensation et espaces de Hardy, *C. R. Acad. Sci. Paris*, 1989, t309, Série I, 945–949.

On viscous compressible flows

12. Lectures on viscous compressible flows given at International Centre for Mathematical Sciences, Edinburgh, November 1995.

Viewing solar mysteries from space

Bhola N. Dwivedi

The solar atmosphere presents a rich tapestry, providing astrophysics with new, unexpected designs of great intricacy. And for millennia we have based our views of the Sun (and the universe) in the narrow visual window of the electromagnetic spectrum. Over the last fifty years or so, this has been extended to the radio, ultraviolet, X-ray, gamma-ray and other parts of the spectrum. This article presents solar mysteries viewed from major space programmes in the past and future space missions underway to unravel these mysteries.

FOR millennia, the Sun (and the universe) has been viewed in the visual light that unaided human eyes are capable of seeing. As the bestower of light and life, the ancients made God out of the Sun. There was consternation, therefore, back in the seventeenth century when Galileo demonstrated that the Sun was not the immaculate object as supposed by ancient philosophers. Since then, interests in our nearest star, the Sun, with its sunspots and related magnetic phenomena (see Figure 1), have been passing gently from the consternation of philosophers to the fascination of astronomers and astrophysicists. Therefore, the observations of the Sun and their interpretations are of universal importance for at least two fundamental reasons. First, our Sun is the source of energy for the whole planetary system including our own planet and almost all aspects of our life have direct relations to what happens on the Sun; and second, our Sun

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