Professor S. N. Bose's contribution to quantum statistics – A critical review

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'The Indian Bose has given a beautiful derivation of Planck's law including the constant (i e. $8\pi\nu^2/c^3$)', wrote Einstein in July 1924 in a letter to Ehrenfest. He further commented that 'Bose's derivation is elegant but the essence remains obscure'[1]. We see that Bose's work, from the very beginning, evoked a sense of mystery which haunted physicists even after a lapse of more than 50 years. The first surprise is that quantum statistics, the last major discovery in quantum theory prior to the advent of quantum mechanics, was initiated by a man almost unknown to the world of physics, living in isolation in Dacca (now in Bangladesh), far removed from the main centres of hectic activities in this field. It is doubtful if there is a second example like this in the history of physics. Perhaps it is this unbelievable element in Bose's discovery which prompted the Nobel laureate scientist Max Delbruck to ask 'Was the Bose–Einstein statistics arrived at by serendipity?'[2]. The same spirit is revealed in Pais' remark that 'Bose's work is the most successful shot in the dark'[3]. There is, however, a more important reason to suspect that the discovery might have been accidental. The first three papers on the subject published in quick succession (the first two papers of Bose[4,5] published in July 1924 and the first paper of Einstein[6] in September 1924) make no mention of the most significant step forward in quantum statistics, viz. the concept of indistinguishability of the particles. It is really surprising that neither Bose nor Einstein in their first paper considered it necessary to examine critically the derivation of the new expression for thermodynamic probability. It is only in his second paper[7] (published in December 1924) that Einstein discussed this problem in detail after Ehrenfest and others pointed out the gap.

In this connection we mention a few more interesting features not quite well known. Bose actually gave two different expressions for thermodynamic probability in his two papers. In the first paper[4] he writes

$$W_1^s = \frac{Z^s\,!}{p_0^s\,!\;p_1^s\,!\;\cdots}\qquad W = \Pi_s W_1^s \qquad \text{(B1)}$$

where Z^s is the total number of elementary phase cells belonging to photons of energy $h\nu^s$, in which N^s photons are distributed. p_r^s is the number of cells which contain r photons. As a result we have

$$\sum_r p_r^s = Z^s \quad\text{and}\quad \sum_r rp_r^s = N^s.$$

In the second paper, Bose writes

$$W_2^s = \frac{(Z^s+N^s-1)\,!}{N^s\,!\,(Z^s-1)\,!} \approx \frac{(Z^s+N^s)\,!}{N^s\,!\,Z^s\,!}$$

$$\text{and}\quad W = \Pi_s W_2^s. \qquad \text{(B2)}$$

Before giving the expression (B1), Bose simply writes 'now it is easy to calculate the thermodynamic probability (macroscopically defined) of a state'. He gave no idea as to how he arrived at the expression or why he chose the particular description of the macroscopic state. In the second paper, while discussing the thermodynamic probability for radiation, Bose writes that 'this has been derived earlier'. And he gives two references. One is a paper by Bose to be published in *Philosophical Magazine*, but actually it was never published. The other is a paper by Debye (the paper published in 1910, which we shall discuss later). Thus, we do not have any record of the details of Bose's derivation of either (B1) or (B2).

Mehra[8], while commenting on the relation between the two expressions, says that W_1^s can be reduced to W_2^s. This, however, is wrong. W_1^s summed over all possible distributions of p_r^s leads to W_2^s. Similar imprecise remarks have also been made by Ghose[9] while comparing (B1) with (B2). Einstein used (B1) in his first paper and derived (B2) using the concept of indistinguishability, in the second paper. Pais[10] remarked wrongly that (B2) is the 'Einstein's expression for W now used in textbooks'. It is generally believed that Bose derived the photon statistics by dropping the constraint on N, the total number of photons. He kept only E, the energy, as constant. This is not quite correct. Such confusions arise because the differences in Bose's two derivations have never been examined critically. In his first paper, it appears that Bose maximized $W = \Pi_s W_1^s$, but in fact he was maximizing W_1^s. Here he was maximizing with respect to variables p_r^s (which are regarded as the macroscopic parameters here) and it turns out that the procedure is equivalent to maximizing each W_1^s separately keeping Z^s and $E^s = N^s h\nu^s$ constant. This implies that the photon number N^s is also kept constant. Pais[11] comments that 'the final irony is that the constraint of fixed Z^s is irrelevant'. This is a careless remark. In the logarithmic derivative of W_1^s, one has to use this condition. However, the additional input of this condition through Lagrange's multiplier turns out to be trivial and unnecessary. But this is only an accidental result. In his second paper, Bose maximized W_2^s with respect to the variables N^s keeping only $E = \Sigma_s N^s h\nu^s$ constant. Many such misconceptions seem to arise because physicists, in general, completely overlook Bose's second paper.

A more significant misconception is that quantum statistics started with Bose's work. In his second paper, Bose starts with a critical examination of a paper by Debye[12] published in 1910, where he used statistical mechanics to derive Planck's law. Bose concludes that Debye's derivation is consistent with the photon description except for his derivation of the factor $8\pi\nu^2/c^3$. Debye used an expression for W^s first written down by Planck in his book *Warmestrahlung* (English translation[13] appeared in 1913). This expression is identical to (B2) but Z^s and N^s were given a slightly different meaning. Natanson[14], in 1911, derived Planck's law using an expression for W^s which is identical to (B1). Both Debye and Natanson described black-body radiation in terms of normal-mode vibrations. This is equivalent to the description of radiation as an ensemble of harmonic oscillators. In 1914, Ehrenfest and Kamerlingh Onnes[15] published a paper in which they tried to interpret Planck's formula (B2) in terms of the distribution of photons. They concluded that if photons are regarded as mutually independent, Planck's formula cannot be obtained (See refs. 8 and 9 for a brief summary of these and other works.)

The influence of these earlier works in the development of quantum statistics has mostly been ignored The central mystery that Bose discovered the photon statistics without discovering the basic concept of indistinguishability of photons remains unresolved. Authors reporting Bose's work fill up the gap by asserting that implicitly Bose introduced the concept of indistinguishability. Such statements are not true and do not put Bose's work in the right perspective.

We suggest an alternative angle to look at Bose's contribution. Based on available facts and some plausible conjectures, we show that Bose was partly right when he said that he was doing a Boltzmann type of calculation. In this work he was strongly influenced by earlier works, particularly Planck and Debye. Thus, a continuity in the development of quantum statistics can be established. This is possible because of a fortunate circumstance. The concept of indistinguishability can be pushed back till we come to discuss the Bose–Einstein statistics of classical particles. For photons this concept is not an essential requirement For example, if one uses Gibbs canonical ensemble formalism, one can avoid the concept of indistinguishability.

There is another interesting aspect of Bose's work. A photon when regarded as a particle has an unusual nonclassical property. It has only one energy state of energy $h\nu$ and momentum $h\nu/c$. If we try to change the momentum, we have to change ν and this will give us a new photon and not an excited state of the earlier photon Thus, for an ensemble of N identical photons of energy $h\nu$, if $E = Nh\nu$ is kept constant, N becomes a constant and no variation is possible. Photons with different frequencies are not indistinguishable If we consider radiation in an enclosure to consist of different frequencies ν^s, then through absorption and re-emission by a speck of matter the number N^s of each type of photons can be changed and variation of the probability function W becomes possible. But such an ensemble consists both of distinguishable and indistinguishable photons. This naturally raises the question, what type of statistical ensemble of photons was considered by Bose? Can this ensemble be compared with the standard Boltzmann or Gibbs ensembles used for classical particles? We discuss this question later.

Bose's two expressions for W: A conjecture on how Bose arrived at them

The basic difference between the two expressions for W which Bose used is in the definition of the thermodynamic macrostate. In (B1) this is defined by the variables p_r^s and in (B2) by N^s. The necessity for this arises because the ensembles considered are really different, though apparently they seem to be the same We shall take up this question later. At present we are interested in the mathematical relation between the two expressions. This relation is rather complex and many misstatements are found in the literature. Let us first write the correct relations:

$$W_2^s = \sum_{(\;p_r^s\;)} W_1^s \quad . \tag{R1}$$

Here s is kept fixed and the summation on the right-hand side is over all possible sets of distributions defined by $(\cdots p_r^s \cdots)$ keeping N^s and Z^s fixed It is evident that $\Pi_s W_1^s \neq \Pi_s W_2^s$ for arbitrary distributions. Let us now suppose that we allow permutations among the photons, regarding them as classical particles. Then we get

$$\sum_{(\;p_r^s\;)} W_1^s \times \frac{N^s!}{\Pi_r (r!)^{p_r^s}} = (Z^s)^{N^s}. \tag{R2}$$

The right-hand side gives the well-known classical value. The correctness of these relations can be checked only by showing that they treat the same combinatorial problem using different procedures. As we see, the relations are far more complex to be comprehended intuitively. It is a good and worthwhile exercise to take up a special case with small values of Z^s and N^s and verify the relations.

In the Planck–Debye derivations of (B2), there is a change of language. They consider radiation as an ensemble of quantized harmonic oscillators. Z^s is interpreted as the number of oscillators of frequency between ν^s and $\nu^s + d\nu^s$. Excitation energy of an oscillator is written as $(n^s + \frac{1}{2})h\nu^s$ and, of all the oscillators in Z^s, $\Sigma n^s h\nu^s$ + constant is the zero-point energy. Dropping the constant part, the total excitation energy of the oscillators is $E^s = N^s h\nu^s$, where $N^s = \Sigma n^s \cdot N^s$ may be defined as the total excitation number for the Z^s oscillators. The difference in the interpretations of the two quantities Z^s and N^s by Bose and Planck–Debye is very significant (Table 1).

It is to be noted that the mathematical expressions for Z^s and E^s are the same in the two interpretations. If we look at the combinatorial problem of distributing the number N^s in Z^s oscillators (Planck–Debye picture) then the result is the same as in (B2). Here Z^s-oscillators are all distinguishable and the question of identity of photons does not arise. Hence, the calculation follows a straightforward Boltzmann method. When we try to evaluate W^s using the Bose picture, two different results are obtained. If we drop permutation among photons, we get (B2). But if we allow permutation, we get $W^s = (Z^S)^{N^s}$. The central problem is that Bose, who was well conversant with Boltzmann's method of calculation, should have written down this expression. Instead he wrote (B2). Why?

In his statement to Mehra[16], Bose stated that he had read Planck's book *Warmestrahlung*. In his second paper[5] Bose examined Debye's paper critically. It is clear that he was familiar with the expression (B2) for W^s. 'Debye has shown that Planck's law can be derived using statistical mechanics', with these words Bose starts his second paper. But Bose

Table 1. Difference in the interpretations of Bose and Planck–Debye

	Bose	Planck–Debye
$Z^s = 8\pi\nu^{s^2} d\nu^s v/h^3$	Number of elementary phase cells, each of volume h^3, belonging to photons of frequency lying in the range ν^s to $\nu^s + d\nu^s$	Number of different harmonic oscillators belonging to the frequency range ν^s to $\nu^s + d\nu^s$
N^s	Total number of photons in Z^s cells	Total excitation number of the Z^s oscillators
$E^s = N^s h\nu^s$	Total energy of all the photons in Z^s	Total energy of all the oscillators in Z^s

objects that his derivation is not completely independent of classical electrodynamics. Because Debye calculates the number Z^s by using the idea of normal-mode vibrations, i.e. using the idea that radiation has a wave nature. Bose strongly believed in Einstein's photon picture of radiation. His one obsession was that if this idea is correct then one must be able to derive Planck's law by considering radiation as an ensemble of particles (photons) and in no way using the wave picture. Debye's use of normal modes in deriving Z^s was considered by Bose as a serious flaw in this scheme. So he replaced this by a new derivation by introducing the concept of phase space of photons each of which has a momentum $h\nu^s/c$. In this new picture Bose writes '$8\pi\nu^2\,d\nu\, v/c^3$ can be interpreted as the number of elementary cells in the six-dimensional phase space of the quanta. *Further calculations (of Debye) remain essentially unchanged*'. The last comment has a crucial significance. It means that Bose assumes that in the photon picture, Debye's calculation of W^s (i.e. (B2)) remains unchanged. By implication it follows that Debye's idea of an oscillator excited to a state of energy $n^s h\nu^s$ is, in Bose's language, equivalent to having n^s phonons each of energy $h\nu^s$. This equivalence is basically right as has been proved after quantization of radiation field. But in 1924 it was not known. Here Bose showed superb intuition. With this equivalence, both the expressions for Z^s and E^s become identical in the two pictures of radiation (ensemble of photons and ensemble of normal-mode vibrations). Hence, if Planck's law is to be derived, the expression for W^s must necessarily be the same in the two pictures. Our conjecture, based on the brief statements of Bose quoted above, is that Bose arrived at this conclusion through intuition and did not wait for a proper analysis of the implications. This conjecture alone can explain the way he introduced the expression for W^s for photons in his second paper – 'Thermodynamic probability for radiation: This has been derived earlier[7]'. Reference 7, as we have mentioned earlier, contains Debye's paper. Our conjecture also explains why Bose thought that he was not introducing any basically new concept in the calculation of W^s. Once the equivalence is taken for granted, one can utilize the Planck–Debye derivation of W^s, which is done entirely in the Boltzmann framework.

What about Bose's formula for W^s in the first paper? We do not know if Bose had seen Natanson's paper of 1911, where the expression (B1) was derived using the Planck–Debye picture. Naturally, here our conjecture will be more speculative. Our guess is that Bose was not aware of Natanson's paper. Otherwise, he would have referred to it as he had referred to Debye in his second paper. Planck, in his book, considered radiation of a single frequency. His ensemble consisted of N^s quantum oscillators all of frequency ν^s. For this he derived the expression for W^s (i.e. (B2)). But here both the parameters Z^s and N^s are constant and there is no variable parameter with respect to which one can maximize W^s. Both Natanson and Bose independently resolved this difficulty by introducing the variable p^s_r. Bose, using photon language, had the additional constraint to make his expression for W^s_1 consistent with the Planck's expression for W^s_2.

After enunciating the combinatorial problem as Bose had done in the first paper, one would immediately write down the expression for W^s_1. The question of permutation of the particles would come later. Our conjecture is that after writing W^s_1, Bose could easily check that it was consistent with W^s_2. So he was sure that the formula was correct, and he overlooked the finer details about the permutation of the particles. One thing seems almost sure, that Bose was not aware of the paper by Ehrenfest and Kamerling Onnes[15]. Had he seen this paper, it seems almost certain that the question of indistinguishability would have taken a prominent part in his paper.

The scenario changed completely when Einstein extended Bose's formula to classical particles. The alternative Planck–Debye picture is no longer valid and one has to justify the derivation of W^s_1 or W^s_2 independently. Hence, the concept of indistinguishability becomes essential.

Let us now summarize our conclusions. Firstly, Bose reinterpreted the Planck–Debye derivation of radiation formula in terms of photon language. In doing this he correctly anticipated that both the languages are true and that the expression for W^s remains unchanged by the change of language. This equivalence made it possible for Bose to derive photon statistics without introducing the concept of indistinguishability. Secondly, Bose introduced the concept of phase space of photons, which made his photon statistics look almost identical to particle statistics. This led to the extension of his formula for W^s to classical particles. In the Planck–Debye language one could not even think of such an extension.

Statistical ensemble in Bose's two derivations

We have already mentioned some very peculiar nonclassical properties of photons. As a result, an ensemble of photons will have properties not exactly similar to those of an ensemble of classical particles. Hence, it is relevant to discuss precisely what sort of ensembles Bose used in his photon statistics.

In the first paper Bose writes $S = k \ln W$, $W = \Pi_s W^s$ (W^s given by (B1)). The constraints are $Z^s = \Sigma_r p^s_r$, $E = \Sigma_s \Sigma_r h\nu^s r p^s_r$. Apparently, it may seem that Bose is considering an ensemble of photons of all frequencies and maximizing W. But, in fact, he was maximizing only W^s keeping Z^s and $E_s = N^s h\nu^s$ constant. It is easy to see that, if we do this, all of Bose's results remain the same. This is because each ensemble of N^s photons has a most probable distribution for p^s_r corresponding to the equilibrium state at a temperature T, say. If all these ensembles are put in thermal contract, they will still be in equilibrium. Thus, maximizing W^s in terms of p^s_r automatically maximizes W.

Thus, Bose, in the first place, was considering a Boltzmann ensemble of N^s identical photons each of one energy stage $h\nu^s$. The nonclassical feature of this ensemble is that when E^s is kept fixed, no variation of W^s is possible. The situation is somewhat like an ensemble of particles all of whose energy states are degenerate. Bose introduced a new definition of a macroscopic state in terms of the parameters p^s_r, which can then be varied to give different values of W^s. It is interesting to note that in terms of the Planck–Debye picture, variables p^s_r come out quite naturally. The ensemble of Z^s oscillators with a total excitation energy $E^s = N^s h\nu^s$ is a simple Boltzmann ensemble, where the distribution parameter is the number of oscillators excited to the rth state of energy $rh\nu^s$, and this exactly corresponds to the variable p^s_r

defined by Bose. This again confirms our conjecture that Bose was intuitively convinced about the mathematical identity of the two descriptions of radiation.

In the second paper, Bose, following Debye, maximizes W and not W^s. Here the distribution parameter is N^s. The ensemble is neither Gibbs canonical nor Boltzmannian. It is semi-Gibbsian. We have N^s identical photons as a subensemble. Varying ν^s, we generate other subensembles and put them in thermal contact. Photons in any two subensembles are nonidentical. Since W^s depends on Z^s and N^s, we can change W^s by varying N^s and then maximize W by the usual technique. The only constraint here is that $E = \Sigma_s N^s h\nu^s$, a constant.

Summary

Bose had been plagued throughout his life by the question how he got the formula for W^s without using the indistinguishability concept. In an interview with Mehra[17] towards the end of his life, he stated: 'I had no idea that what I had done was really novel . . . Instead of thinking of the light quantum just as particles, I talked about these states. Somehow, this was the same question which Einstein asked when I met him (in October or November 1925), how had I arrived at this method of deriving Planck's formula.' Bose thought that he was using the standard Boltzmann method. And we believe that he was right.

The development of quantum statistics is not as abrupt as is usually believed. There is a continuity starting from the works of Planck, Debye, Natanson, Ehrenfest and Kamerling Onnes and others. This has been possible because, for radiation, there are two alternative ways of treatment, one may consider it as an ensemble of distinguishable quantized harmonic oscillators (Planck–Debye picture) or as an ensemble of indistinguishable photons (Bose picture). Both lead to the same expression for W. Bose used the photon language to describe the ensemble but the Planck–Debye picture to calculate W and thereby sidetracked the question of indistinguishability. When Einstein extended Bose's expression for W to particles, then the alternative Boltzmann picture of oscillators disappeared. One can derive the particular expression for W by using the concept of indistinguishability alone. And this Einstein did in his second paper.

In the history of physics, the discovery of the B–E statistics is a remarkable example to show how tortuous the path along which ideas in physics develop is. We are reminded of the incisive remark by Koestler[18], made in connection with the discoveries in cosmology: 'The history of cosmic theories may without exaggeration be called a history of collective obsessions and controlled schizophrenia; and the manner in which some of the most important individual discoveries were arrived at reminds one more of a sleep-walker's performance than an electronic brain's.'

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