selves are reduced to their single principle. There is thus an incessant multiplication of the inexhaustible One and unification of the indefinitely Many. Such are the beginnings and endings of worlds and of individual beings: expanded from a point without position or dimensions and a now without date or duration, accomplishing their destiny, and when their time is up returning 'home' to the Sea in which their life originated.

[A] Augustine, St, Concerning the nature of good, in *Basic Writings of St Augustine* (ed. Oates, W J), Random House, New York, 1948, vol I

[C1] Coomaraswamy, A. K, *Hinduism and Buddhism*, Philosophical Library, New York (n d)

[C2] Coomaraswamy, A K., *Spiritual Authority and Temporal Power in the Indian Theory of Government*, American Oriental Society, New Haven, Conn, 1942

[E] Edgerton, F, *The Bhagavad Gita*, volume I, Harvard University Press, Cambridge, Mass, 1946.

[J] James, William (1897), *The Will to Believe and Essays in Popular Philosophy*, Dover, New York, 1956.

[K] Kangle, R. P., *The Kautiliya Arthasastra*, Part II, Motilal Banarsidass, Delhi.

[M] Masani, P. R., *Norbert Wiener: 1894–1964*, Vitae Mathematica, Birkhauser, Basel, 1990, vol. 5.

[P] Perice, Charles S., *Collected Papers* (eds Hartshorene, C., Weiss, P. and Burks, A), Harvard University Press, Cambridge, Mass., 1960.

[W] Whitehead, A. N., *Science and the Modern World*, Cambridge University Press, Cambridge, 1933.

*In the Wiener references, numbers in square brackets refer to the bibliography adopted in his *Collected Works*, and the Roman numerals refer to the volume in which the paper is reprinted.

[14d] Wiener, N, Relativism, *J. Phil. Psych and Sci. Method*, 1914, **11**, 561–577. I

[28d] Wiener, Norbert, Coherency matrices and quantum theory, *J Math. and Phys.*, 1928, **7**, 109–125. III

[VII] Wiener, N, [Defense Department Documents], A A. Directors, Summary Report of Demonstration (17p), June 10, 1942

[XII] Wiener, N, [Defense Department Documents], Automatic Control Techniques in Industry, Industrial College of the Armed Forces, Washington, D C., 1952–1953.

[49g] Wiener, Norbert, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications*, The MIT Press, Cambridge, Mass, Wiley, New York; Chapman & Hall, London, 1949

[50j] Wiener, Norbert, *The Human Use of Human Beings*, Houghton Mifflin, Boston, 1950, De Capa Series in Science (paperback), Plenum, New York, 1988

[53h] Wiener, Norbert, *Ex-prodigy: My Childhood and Youth*, Simon and Schuster, New York, 1953, The MIT Press, Cambridge, Mass, (paperback) The MIT Press, 1965

[56g] Wiener, Norbert, *I Am a Mathematician The Later Life of a Prodigy*, Doubleday, Garden City, New York, 1956

[60a] Wiener, Norbert, An application of physics to medicine, in *Medicine and Other Disciplines*, New York Academy of Medicine, Galderston, International Universities Press, 1960, ed 1, 41–57 IV

[60e] Wiener, Norbert, The duty of the intellectual, *Tech Rev*, 1960, **62**, 26–27 IV

[61c] Wiener, Norbert, *Cybernetics*, Second edition of [48f] (revisions and two additional chapter), The MIT Press and Wiley, New York, 1961; paperback edition, The MIT Press, 1965.

[62c] Wiener, Norbert, *Short-time and long-time planning*, originally presented at 1954 ASPO National Planning Conference. Jersey Plans, An ASPO Anthology (1962), 29–36 IV

The mathematical work of N. Wiener (1894–1964)

S. D. Chatterji

Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne

NORBERT Wiener's intellectual activity coursed through multifarious channels; beginning with logic and philosophy, it surged through pure mathematics, mathematical physics, engineering and statistics touching on its way literary, political and social criticism. Today, Wiener is widely remembered in the general scientific audience as a founder of cybernetics. This popular appreciation of Wiener's work is not entirely wrong; however, to mathematicians, Wiener's name is always associated with fundamental advances in twentieth century analysis. If Wiener had written nothing else other than his mathematical papers, his name would still remain alive amongst modern mathematicians – even though, admittedly, his popular impact would have been much less.

The purpose of the present essay is to describe some parts of Wiener's significant contributions to mathematics in as simple and non-technical a language as possible. More detailed and technical presentations can be found in items [A], [C], [M] of the short list of references given at the end of this article; a brief chronology of Wiener's life is given there as well. References to Wiener's original papers will be given via his collected works [W]; an excellent source for acquiring further understanding of all aspects of Wiener's work is the biography [M] by Masani.

Although Wiener was extremely precocious in his intellectual development (as would be evident from even a cursory glance at the chronology given at the end), his best-known mathematical contributions were mostly the product of his work during the ages of 25 to 50 (1919–1944), as is the case of a vast majority of mathematicians. If we were to select one single predominant characteristic of Wiener's mathematical work, it would be the skilfull utilization of Lebesgue type integration theory; naturally, this was combined with other techniques, most notably, from complex function theory; however, it is the early and sustained use of modern integration theory that seems to be a hall-mark of Wiener's mathematical art. Wiener himself has emphasized this aspect of his work at several places and has indicated, in particular, the important role that Lebesgue's integration theory has played in his own interpretation and elaboration of the probabilistic ideas contained in Gibbs' formulation of statistical mechanics. Let us recall briefly how the original Lebesgue theory of integration in the Euclidean spaces $\mathbb{R}^N$ had matured, around 1910–1918, into a very general theory applicable in abstract spaces. Following some early considerations of Borel, Lebesgue's first definitive work appears in his thesis of 1902. Lebesgue's considerations were restricted to studying $\int f(x)\,\mathrm{d}x$ for $x \in \mathbb{R}^N$; Radon, in a long paper published in 1913 extended the study to (Stieltjes) integrals of the type $\int f(x)\,\mu(\mathrm{d}x)$ where μ is a general mass or charge distribution in $\mathbb{R}^N$; W. H. Young (who had independently discovered a Lebesgue type integral around 1904) introduced a new method for *studying the Lebesgue–Radon–Stieltjes integrals (still in* $\mathbb{R}^N$) using monotone sequence of (semi-)continuous functions. Young's methods led P. J. Daniell to formulate a general theory in 1918 (which is still called the Daniell theory or Bourbaki–Daniell–Stone theory, after the names of subsequent authors who completed and popularized Daniell's work). Needless to say, the above is a very fore-shortened version of the actual picture concerning the development of integration theory during 1900–1918; it neglects important issues and contributions associated with the names of Carathéodory, Fréchet, Fubini, Levi, Riesz, Vitali and many others; the interested reader will find a mathematically reliable description of some of this history in Bourbaki's chapter on integration in [B]. The point of the preceding brief historical sketch is to indicate the state of affairs concerning integration theory which faced the youthful Wiener in 1919–1920 as he began to apply Daniells' theory to two entirely different areas of mathematics – one classical, viz. potential theory (during 1923–1925) and the other, an as yet non-existent area in a strictly mathematical sense, viz. probability theory in function spaces. Wiener's first paper in the latter area appeared in 1920 ('The mean of a functional of arbitrary elements', item [20c] in vol. 1 of [W]) followed by a series of others during the following years culminating in his famous paper of 1923 ('Differential-space', item [23d] in vol. 1 of [W]) in which Wiener gave, amongst many other things, a mathematical construction of an important probability measure on C([0, 1]), the space of all continuous, real functions on [0, 1]; this is now widely known as the Wiener measure and it provides the precise framework for a mathematical study of the Brownian motion; indeed, it has remained the basic paradigm for the study of all continuous stochastic processes. We may add here that the Wiener measure and the associated stochastic process were to permeate a large part of Wiener's thinking for the rest of his life and a great deal of his mathematical (and other) work flows, directly or indirectly, from the considerations of his 1923 paper; this is eloquently summarized in the following statement of Wiener himself, taken from his autobiographical volume *Ex-prodigy* (1953), pp. 274–275: 'Most of my later work in mathematics goes back in one way or another to my study of the Brownian motion. In the first place, this study introduced me to the theory of probability. Moreover, it led me very directly to the periodogram, and to the study of harmonic analysis more general than the classical Fourier series and Fourier integrals. All these concepts have combined with the engineering preoccupations of a professor of the Massachusetts Institute of Technology to lead me to make both theoretical and practical advances in the theory of communication, and ultimately to found the discipline of Cybernetics, which is in essence a *statistical approach to the theory of communication.* Thus, varied as my scientific interests seem to be, there has been a single thread connecting them all from my first mature work to the present.' Remarks of a similar nature appear scattered elsewhere in Wiener's writings (e.g. in *I am a mathematician*, *1956*, *Wiener's second* autobiographical volume) and we shall try to show in this article how well justified Wiener's self-analysis was; we shall however restrict ourselves only to his mathematical work, although the remarks contained in the above citation apply to a great deal of his other contributions.

It is to be noted at this point that by 1919 Wiener had mastered the standard theory of Lebesgue integrals very thoroughly, in particular, during his studies in Cambridge (England) in 1913–1914 where he enthusiastically followed a course of analysis from G. H. Hardy containing 'the first principles of mathematical logic, by way of the theory of assemblages, the theory of the Lebesgue integral, and the general theory of functions of a real variable, to the theorem of Cauchy and to an acceptable logical basis for the theory of functions of a complex variable.... If I am to claim any man as my master in my mathematical training, it must be G. H. Hardy' (*Ex-prodigy*, p. 190).

We shall now try to explain in what way the Daniell generalization of Lebesgue's integration theory enters into Wiener's work. First however we must spend a little time in explaining briefly the nature of the Lebesgue integral and in what way this integral was generalized by Daniell and others.

The integration of a continuous function $f:\mathbb{R}^N\to\mathbb{R}$ with compact support (i.e. $f(x)$ vanishes for all x outside a bounded set) was a well-understood subject by the middle of the 19th century. The case $N=1$ and $f\geq 0$ concerns the determination of the area under the curve defined by the equation $y=f(x)$ and in this form it is of great antiquity dating back to studies of Archimedes (3rd century BC); its more systematic development by Leibniz and Newton in the 17th century was part of the development of the infinitesimal calculus; generalization to $N\geq 2$ along with numerous further developments was worked out by the Bernoullis and Euler during the 18th century (with the mathematical rigour prevalent in that period); with Cauchy's work in the early 19th century the logical foundation of the $\int f(x)\,dx$ (f as above) may be said to be complete, at least in the one dimensional case ($N=1$) – although functions of more than one variable continued to cause difficulty to many mathematicians, even famous ones, until very late in the 19th century. The definition adopted was that

$$\int f=\int f(x)\,dx=\lim\sum_{i=1}^{n}f(x_i)m(A_i),$$

where $A_1, A_2, \ldots, A_n$ is a finite family of non-overlapping rectangular sets (N-dimensional rectangular parallelopipeds) whose union contains the bounded set where $f\neq 0$, $x_i\in A_i$, $1\leq i\leq n$, and $m(A_i)$ is the N-dimensional volume of A_i (i.e. $m(A_i)$ equals the product of the N lengths of the N 'sides' of A_i); the limit is taken in the sense that all the sides of each of the A_i tend to 0; this last point must be defined more carefully but this is now done in all rigorous courses of first year university calculus and will not be further discussed here. Naturally, the existence of the limit needs proof; this and other facts (for f continuous with bounded support) can be considered to have been well-established by the 1850s. One problem with the above theory was that often one needed to integrate functions f which were not continuous; when f had simple discontinuities the definition of $\int f$ could be re-adjusted in a satisfactory but ad hoc manner; however, this ad hoc cobbling was not found sufficient. Riemann, in his 1854 Habilitationsschrift (published after his death in 1867; such a Habilitationsschrift in German universities is written after a doctoral dissertation – Riemann wrote his doctoral dissertation in 1851 – in order to obtain a 'venia legendi', the permission to teach as a Privatdocent) on trigonometrical series, had given, in passing, as it were, the definition of $\int f$ (for the case $N=1$) for an arbitrary bounded function f with bounded support; Riemann's main interest in writing his Habilitationsschrift lay elsewhere and he did not study the integral in any detail except to give a necessary and sufficient condition for the existence of $\int f$ as a limit of sums of the type (1) (known now as Riemann sums associated with f). Of the several drawbacks of Riemann integrability let us mention the following: suppose that $f(x)=\lim_{n\to\infty}f_n(x), x\in\mathbb{R}^N$ with $f_n:\mathbb{R}^N\to\mathbb{R}$, $n=1,2,\ldots,$ uniformly bounded and vanishing outside a fixed bounded set; even if all the f_n's are Riemann integrable, f need not be so. This shortcoming of Riemann integrability is removed in Lebesgue's theory which however we shall not try to describe here since it would necessarily be rather lengthy. Instead, we shall indicate Daniell's approach which is more general and which is the one that Wiener uses in his work.

In Daniell's theory, we start with a vector space T_0 of bounded real-valued functions defined on any set X; it is supposed that if $f\in T_0$ then $|f|\in T_0$ (whence follows that T_0 is a vector-lattice, i.e. $f, g\in T_0$ implies that $\max(f, g)$ and $\min(f, g)$ are also in T_0); further, let the constant functions belong to T_0. (Some of the hypotheses on T_0 can be relaxed at the cost of some added complications in the description of the theory; the structure imposed on T_0 above will be sufficient for our further developments.) Let $I:T_0\to\mathbb{R}$ be a linear functional which is positive (i.e. $f\geq 0\Rightarrow I(f)\geq 0$) and which verifies the property (L): if $\{f_n\}$ is a decreasing sequence of functions in T_0 such that $f_n\to 0$ as $n\to\infty$ then $I(f_n)\to 0$ as $n\to\infty$. Daniell's main theorem is then that I can be extended to a larger vector-lattice of functions $T\supset T_0$ in such a way that I remains a positive linear functional having the property (L) and the following two further properties: (1) (monotone convergence property) if $f_1\leq f_2\leq\ldots$ is a sequence of functions in T and $\lim_n I(f_n)$ is finite then $f=\lim_n f_n$ is in T; (2) (the dominated convergence property) if $f_1, f_2,\ldots$ is any sequence of functions in T such that $f_n\to f$ as $n\to\infty$ and $|f_n|\leq g$, $n\geq 1$, for some $g\in T$, then $f\in T$ and $I(f_n)\to I(f)$ as $n\to\infty$. The functions in T are called *summable functions*; it turns out that there is a countably additive positive measure μ on a σ-algebra of subsets Σ in X such that T is made up of exactly the μ-integrable real functions on X and that for any $f\in T$, $I(f)=\int f\,d\mu$. This last point was not established by Daniell although the idea of considering integrability of function on an abstract measure space (X, Σ, μ) had been studied in detail earlier by Fréchet in a paper published in 1915; this is now standard material for probability or integration theory courses and will not be explained any further here. What I find interesting to add at this point is the following historical remark: Wiener who was familiar with many of Fréchet's writings of this period and who had even known Fréchet well around 1920 did not attempt to correlate Fréchet's 1915 work on integration theory with that of Daniell; neither did Daniell try to establish any connection between his theory and that of Fréchet whose 1915 paper he refers to in his own 1918 article. In fact, the relationship mentioned above between Daniell's summable functions and Fréchet's integrable functions is not difficult to establish and the proofs known to-day use very much the type of techniques that Daniell was using in his paper. This clarification would have been useful to Daniell as well as to Wiener; in particular, Wiener needed at various points of his work that the indicator functions of certain sets were indeed summable and, in each case, he uses an ad hoc argument to establish this. Wiener however seemed to be quite familiar with the special cases where Daniell's summable functions coincide with Lebesgue–Radon–Young–Stieltjes integrable functions: here X is a bounded, closed rectangular set in $\mathbb{R}^N$, T_0 consists of the continuous real-valued functions defined on X and $I(f)=\int_X f\,dm$ in the sense of equation (1) where m is either the ordinary volume of rectangular sets (as above) or some other mass or charge distribution; if m is the ordinary volume, Daniell's summable functions are exactly the Lebesgue integrable functions in X; for a more general distribution m one obtains the so-called Lebesgue–Stieltjes integrable functions; this was clear from the previous studies of Radon and Young, and Wiener was thoroughly aware of this since he used this fact at several places of his six papers devoted to potential theory published during the years 1923–1925 (all reproduced in [W] vol. 1). We shall start with a brief description of the nature of Wiener's contributions in these potential theory papers; all of Wiener's results in these papers have been thoroughly integrated into modern potential theory which has been perfected to such a great degree due to the works of Brelot, Choquet, Deny and many of their brilliant followers during the last 50 years. Wiener never returned to this field later, although potential theory was to play such a central role in the development of probability theory during the fifties and sixties through the work of Doob and Kakutani on continuous Markovian stochastic processes related to Brownian motion.

The central problems of classical potential theory are related to the solution of the Dirichlet problem: given a domain (connected, open set) $D\subset\mathbb{R}^N$ and a continuous real-valued function φ defined on Fr D (the frontier of D; thus $\varphi\in C(\mathrm{Fr}\,D)$) the Dirichlet problem consists in the determination of a continuous real-valued function u defined on $\bar{D}=D\cup\mathrm{Fr}\,D$ which is twice continuously differentiable in D and such that $u=\varphi$ on Fr D and $\Delta u=0$ (where $\Delta u=\partial^2u/\partial x_1^2+\cdots+\partial^2u/\partial x_N^2$ is the Laplacian of u). If the domain D is bounded, it was known for a long time, that the solution to the Dirichlet problem, if it existed at all, was unique; further, various examples of bounded domains were known where a solution did not exist and for many types of relatively 'regular' bounded domains, not only the existence of the solution was known but one knew various formulae for representing them. It was in this context that Wiener tried to 'solve' the Dirichlet problem for arbitrary bounded domains. One of his starting points is the following observation (cf. [23c] in [W], vol. 1): suppose that D is a bounded domain in $\mathbb{R}^N$ which is *regular* for the Dirichlet problem in the sense that there is a (unique) solution $u=u_\varphi$ corresponding to any specification of a continuous boundary value φ (in the notation above); then, to each $y\in D$, there exists a probability measure μ_y defined on the (Borel) subsets of Fr D such that

$$u_\varphi(y)=\int_{\mathrm{Fr}\,D}\varphi(x)\mu_y(\mathrm{d}x). \tag{2}$$

The measure μ_y is called the harmonic measure on Fr D and the simple proof of its existence given by Wiener is quite modern in spirit (and actually the one generally used today). It goes as follows: for each $y\in D$ define I_y on $T_0=C(\mathrm{Fr}\,D)$ by the formula

$$I_y\varphi=u_\varphi(y),\quad \varphi\in T_0;$$

then it is easy to verify that I_y is a positive, linear functional satisfying Daniell's limit condition (L); thus I_y corresponds to integration with respect to a positive measure μ_y defined on the (Borel) subsets of Fr D; since $I_y(1)=1$ it is clear that each μ_y is a probability measure. Wiener then goes on to prove that if f is any real-valued function defined on Fr D which is μ_y-integrable for some $y\in D$ then it is μ_y-integrable for all $y\in D$ and the function defined by

$$v_f(y)=\int_{\mathrm{Fr}\,D}f(x)\mu_y(\mathrm{d}x),\quad y\in D$$

defines a function harmonic in D (i.e. $\Delta v_f=0$ in D); Wiener then shows that if f is bounded and is, furthermore, continuous at a point $p\in\mathrm{Fr}\,D$ then $\lim_{y\to p}v_f(y)=f(p)$. In a subsequent paper ([24a] in [W], vol. 1) Wiener showed how to associate a harmonic function u_φ to each φ in $C(\mathrm{Fr}\,D)$ where D is an *arbitrary* bounded domain in such a way that the 'boundary values' of u_φ are related to φ in a reasonable manner; this is done by approximating D from the inside by a sequence of regular domains $D_n\subset D$. In a later paper ([25]a in [W], vol. 1) Wiener showed that his solution u_φ is the same as another one given by Perron as well as Remak (independently of Wiener, in 1923 and 1924 respectively). Wiener goes on to introduce the important notion of the capacity of an arbitrary set and uses this to give a sufficient condition that a point $p\in\mathrm{Fr}\,D$ be a *regular* point in the sense that $u_\varphi(x)\to\varphi(p)$ as $x\to p$ for all $\varphi\in C(\mathrm{Fr}\,D)$) where u_φ is his '*generalized*' solution associated with the boundary function φ. Thus, Wiener in his six papers during 1923–1925 introduced and studied in great generality some of the central issues of potential theory: generalized solutions, capacity of sets and regularity of boundary points.

We now describe Wiener's work on the theory of Brownian motion, one of his most fruitful and important creations in mathematics. Let ξ_t denote, say, the x-coordinate of the position of a particle in Brownian motion at time t; let $\xi_0 = 0$; it was known and very clearly argued in a celebrated 1905 article of Einstein on Brownian motion that the random variable ξ_t will have a normal (Gaussian) distribution with mean 0 and variance $\sigma^2 t$ (briefly, ξ_t is $N(0;\ \sigma^2 t)$) where $\sigma > 0$ depends on various physical characteristics of the fluid, the particle and some universal physical constants; further, it was natural to suppose that if $0 \le s < t$, $\xi_t - \xi_s$ will be independent of all the positions ξ_u, $u \le s$ and that $\xi_t - \xi_s$ will be distributed as $N(0,\ \sigma^2(t-s))$. The problem facing mathematicians was to establish the existence of a suitable mathematical edifice which would accommodate a family of random variables ξ_t with the above structure Since the physically observed paths $t \mapsto \xi_t$ of a particle under Brownian motion were continuous, albeit very irregular, it was natural to try to model ξ_t (with t, say, restricted to [0, 1]) as the value at t of a typical element x of the space $X = C_0([0,\ 1])$ consisting of those continuous, real-valued functions on [0, 1] which vanish at 0. Thus, Wiener formulated the problem as one of constructing a probability measure W defined on a suitable family of subsets of $C_0([0,\ 1])$ which has the following properties (here $\sigma = 1$ for simplicity):

- for $0 \le s < t \le 1$ and any bounded (Borel) function $f: \mathbb{R} \to \mathbb{R}$

$$\int_X f(x(t) - x(s))\mathrm{d}W(x) = \int_{-\infty}^{\infty} f(u)p(u; t-s)\mathrm{d}u$$

- if $0 < t_1 < t_2 < \cdots < t_n \le 1$ and $f: \mathbb{R}^N \to \mathbb{R}$ is any bounded (Borel) function of n real variables then

$$\int_X f(x(t_1), \ldots,\ x(t_n))\,\mathrm{d}W(x) = \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\,p(u_1; t_1)$$

$$p(u_2 - u_1; t_2 - t_1) \ldots p(u_n - u_{n-1}, t_n - t_{n-1})\mathrm{d}u_1 \ldots \mathrm{d}u_n. \quad (3)$$

If now T_0 is taken to be the space of all bounded functions F on $X = C_0([0,\ 1])$ which 'depend only on finitely many time points' (i.e. $F(x) = f(x(t_1), \ldots, x(t_n))$ for some choice of $t_1, \ldots, t_n$, $0 < t_1 < t_2 < \ldots < t_n = 1$ and f a bounded (Borel) function of n real variables) and $I(F)$ is defined by equation (3) then it is clear that $I: X \to \mathbb{R}$ is a positive, linear functional as in Daniell's theory (along with the further property that $I(1) = 1$). It is however not at all clear that I satisfies the limit condition (L); Wiener established (L) by an interesting 'tightness' argument which we shall not go into here; suffice it to say that this tightness argument plays a central role in many modern discussions of integration theory in infinite dimensional spaces. Needless to say Wiener's presentation in his famous 'Differential-space' paper of 1923 (referred to above) is not quite as direct as perhaps implied by the preceding lines. Nonetheless this 1923 paper is generally acclaimed to be the first rigorous study of the Brownian motion; this paper (along with six other papers published during 1920–1924, all contained in [W], vol. 1) is remarkable for the pregnancy of the mathematical ideas contained therein, even though Wiener's heuristic exposition makes its conscientious reading quite laborious.

We shall now briefly outline how Wiener's work on generalized harmonic analysis and Tauberian theorems flowed from his study of the Brownian motion. For this purpose, let us consider a Brownian motion $\{\xi_t\}$ where t now ranges over $-\infty$ to ∞; from the point of view presented above this involves defining a probability measure W on $C(\mathbb{R})$, the space of all real-valued continuous functions defined on $\mathbb{R}$, such that if $\xi_t(x) = x(t)$, $x \in C(\mathbb{R})$, $t \in \mathbb{R}$ then ξ_t is $N(0,\ |t|)$ and $(\xi_t - \xi_s)$ is independent of all ξ_α, $\alpha \le s$, for all $s < t$. The existence of such a measure W can be proven in much the same way as above (or else deduced from the measure constructed above). Now Wiener had proven in his 1923 paper that with W-probability one, the Brownian paths $t \mapsto \xi_t$ (i.e. a set of x's in $C([0,\ 1])$ of W-probability 1), although continuous, were extremely irregular; they were of unbounded variation, nowhere differentiable, but of Hölder-class $\alpha < 1/2$. Wiener had also perceived that the fluctuations of a Brownian path were very similar to that of the voltage difference often observed between two terminals and which was known as the 'shot effect'; superposed on electrical signals this shot effect was a noise which tended to distort the original signal. He was thus motivated to extend the classical Fourier or harmonic analysis to such irregular functions. One of his remarkable innovations was the association of a 'generalized Fourier transform' to a large class of functions f which were not necessarily in the classical $L^2(\mathbb{R})$-class and which included almost all Brownian paths (and all almost periodic functions); the condition imposed on f was that f should have an auto-correlation function φ:

$$\varphi(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(x)}\,\mathrm{d}x, \quad t \in \mathbb{R}.$$

Note that if f is in $L^2(\mathbb{R})$ then $\varphi \equiv 0$ and that if $\varphi(0)$ exists then

$$\sup_{T>0} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2\,\mathrm{d}x < \infty$$

which implies (quite easily) that

$$\int_{-1}^{1}|f(x)|^2 dx < \infty, \quad \int_{|x|>1}|f(x)/x|^2 dx < \infty;$$

this last fact is generally indicated by stating that

$$\int_{-\infty}^{\infty}|f(x)|^2/(1+x^2)dx < \infty.$$

If we define (following Wiener)

$$s_1(u) = \frac{1}{\sqrt{2\pi}} \int_{|x|>1} \frac{f(x)e^{-iux}}{-ix} dx,$$

$$s_2(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x)\frac{e^{-iux}-1}{-ix} dx$$

and $s = s_1 + s_2$ then s is called (by Wiener and his followers) *the generalized Fourier transform* of f; the function s_1 exists (at least almost everywhere) since it is the L^2-Fourier transform of the function f_1 which equals $f(x)/(-ix)$ if $|x|>1$ and is 0, if $|x|\le 1$, and $f_1 \in L^2(\mathbb{R})$; the function f_2 which equals $f(x)$ if $|x|\le 1$ and is 0 if $|x|>1$ is square integrable and hence integrable (since f_2 vanishes outside the compact interval [−1, 1]). Recall that if g is in $L^2(\mathbb{R})$ then its L^2-Fourier transform $\hat{g}$ satisfies the following: $\hat{g}(u) = 1/\sqrt{2\pi}\int_{-\infty}^{\infty} g(y)e^{-iuy}dy$ (in the L^2 sense) and

$$\int_{0}^{x} \hat{g}(u)\,du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\frac{e^{-ixy}-1}{-iy} dy.$$

It is thus clear that the 'derivative' of the generalized Fourier transform s of f (as defined by Wiener) is the 'genuine' Fourier transform of f (considered as a tempered distribution) in the sense of Schwartz; since this point has not been clearly stated in the literature, let us briefly outline the argument. If ψ is any rapidly decreasing C^∞-function on the line (of the Schwartz class $\mathbb{S}$) then we shall verify that

$$\int_{-\infty}^{\infty} f(x)\hat{\psi}(x)dx = -\int_{-\infty}^{\infty} s(x)\psi'(x)dx = \int_{-\infty}^{\infty} s'(x)\psi(x)dx$$

which means exactly, that the tempered distribution $\hat{f}$ (defined by $\langle \hat{f}, \psi\rangle = \langle f, \hat{\psi}\rangle$, $\psi \in \mathbb{S}$) is the distributional derivative of s; the verification is contained in the following easily justifiable formulae:

$$\int_{-\infty}^{\infty} f(x)\hat{\psi}(x)dx = -\int_{-\infty}^{\infty} f_1(x)\{ix\hat{\psi}(x)\}dx + \int_{-\infty}^{\infty} f_2(x)\hat{\psi}(x)dx$$

$$= -\int_{-\infty}^{\infty} f_1(x)(\psi')^{\wedge}(x)dx + \int_{-\infty}^{\infty} \hat{f}_2(x)\psi(x)dx$$

$$= -\int_{-\infty}^{\infty} \hat{f}_1(x)\psi'(x)dx - \int_{-\infty}^{\infty} s_2(x)\psi'(x)dx$$

(since $s_2' = \hat{f}_2$, at least, in the distributional sense)

$$= -\int_{-\infty}^{\infty} s_1(x)\psi'(x)dx - \int_{-\infty}^{\infty} s_2(x)\psi'(x)dx = -\int_{-\infty}^{\infty} s(x)\psi'(x)dx$$

(since $s_1 = \hat{f}_1$ and $s = s_1 + s_2$).

It now becomes clear, in analogy with the classical Parseval–Plancherel formula, why Wiener tries to prove that the 'L^2-norm of s'' equals the 'L^2-norm of f' in the following form:

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(x+\varepsilon)-s(x-\varepsilon)|^2 dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2 dx. \quad (4)$$

Formula (4) is an important formula in Wiener's theory; its proof remains difficult even today. To prove (4) Wiener was eventually led to the development of a so-called Tauberian theorem which in its turn led to the profound Wiener Tauberian theory which, at one stroke, generalized, on the one hand, a wide variety of deep, classical theorems due to Hardy, Littlewood, Tauber and others and, on the other hand, has given rise to a broad spectrum of methods and problems in modern harmonic analysis in locally compact groups. In his autobiographical volume *I am a Mathematician* (p. 115) referred to above, Wiener describes how he was led to this work through a chance conversation with Ingham. Space does not permit me to describe here Wiener's work on Tauberian theory, one of his most brilliant contributions to modern analysis; however, many books in harmonic analysis give an introduction to Wiener's Tauberian theory (e.g. Katznelson, *Introduction to Harmonic Analysis*, 1968) as does Wiener himself in his famous book *The Fourier Integral and Certain of its Applications*, 1933; this last describes Wiener's generalized harmonic analysis as well.

In the above survey, I have been obliged to leave out two important areas of mathematics where Wiener made fundamental contributions. The first concerns certain aspects of the classical theory of analytic functions evoked by the appellation 'Paley–Wiener theory'; R.E.A.C. Paley (1907–1933), an English mathematician, collaborated with Wiener during 1932–1933; an account of his work is contained in their joint monograph *Fourier Transforms in the Complex Domain* (1934) and in many other modern texts. The second area involves Wiener's work on prediction and filter theory, carried out, in part, in collaboration with engineers, scientists and other mathematicians during 1940–1959; this also gave rise to several important monographs: *Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications* (1949); *Nonlinear Problems in Random Theory* (1958); a partly non-mathematical work *Cybernetics* (1948, 1961) which lays out the philosophical and mathematical point of view of Wiener concerning the field of Cybernetics; the mathematics invoked in this last book touches on most of Wiener's previous mathematical work on probability theory and analysis and gives us an elegant essay which eloquently formulates a great deal of Wienerian Weltanschauung.

Lastly, it must be added that there are many important mathematical results or insights due to Wiener which have not been adumbrated at all in the above. We should at least mention his important ergodic theorems, his attempted applications of these theorems to turbulence ('homogeneous chaos theory'), the Hopf–Wiener theory for certain integral equations as well as several elegant results of his in general Fourier analysis (e.g. concerning absolutely convergent Fourier series) and their applications to number theory (e.g. Ikehara–Wiener Tauberian theorem and the prime number theorem). And finally, in order to underline again, the numerous other contributions of Wiener, we must reiterate that the present article does not even attempt to touch on the copious writings of Wiener in fields like physics, engineering, etc. where the mathematical component of Wiener's work is far from being negligible. A perusal of the collected works of Wiener leads us to appreciate the fundamental unity in Wiener's creation as well as to the reflection that here is an immense amount of mathematical suggestion which awaits to be developed by the future generation.

Brief chronology of Norbert Wiener's life

1894	Born on Nov. 26 in Columbia, Missouri, USA
1909	Received an A.B. degree from Tufts College (near Boston, USA)
1913	Received a PhD degree in philosophy (with a dissertation in mathematical logic) from Harvard University
1913–1916	Postgraduate studies at the universities of Cambridge (England), Göttingen, Columbia (USA)
1919	Appointed instructor of mathematics at MIT
1926–1927	Marriage; research and teaching in Göttingen and Copenhagen
1931–1932	Visiting lecturer at Cambridge University (England); appointed professor of mathematics at MIT
1935–1936	Visiting professor in Peking (China)
1940–1945	Wartime research leading to prediction and filter theory
1946–1964	Numerous lectures at various institutions in the USA, Mexico and overseas (Europe, India, Japan)
1964	Died of heart attack on 18 March in Stockholm, Sweden.

[A] *Bulletin of the American Mathematical Society*, vol 72, January, 1966, special issue entitled Norbert Wiener 1894–1964, 145 pages

[B] Bourbaki, N., *Eléments d'histoire des mathématiques*, Masson, Paris, 1984 (cf the three chapters on the history of integration theory, pp 275–308).

[C] Chatterji, S. D, *Life and Works of Norbert Wiener* (1894–1964), Jahrbuch Überblicke Mathematik, Vieweg Verlag, Wiesbaden, 1993, pp. 153–184

[M] Masani, P., *Norbert Wiener 1894–1964*, Birkhauser Verlag, Basel, 1990

[W] *Collected works of Norbert Wiener* (ed Masani, P), MIT Press, Cambridge, Mass, vol. I, 1976, vol II, 1979; vol III, 1981; vol IV, 1985