Causes of the failure of Rayleigh–Jeans radiation formula as speculated by some physicists and the ultimate answer from Einstein

Amar Nath Nigam

The radiation formula derived by Rayleigh and Jeans was based on firm classical foundations but it failed to account for the observed distribution of intensity in the thermal spectrum of black body radiation. This led the then leading physicists to look for the cause of this unexpected failure. These attempts due to Rayleigh, Jeans, Lorentz and Ritz are first described. Finally Einstein's explanation is discussed giving conceptual details but avoiding detailed mathematical steps. The ultimate cause lay in the nature of fluctuations in the radiation momentum which is different for long and short wavelength regions. The derivation of Rayleigh–Jeans formula is valid only for the long wavelengths.

Nothing succeeds like success. The works of Planck and Einstein were the subjects of review by several authors. However at the base of this success lies a chain of failures. These too were attempts by leading physicists like Rayleigh, Jeans, Lorentz, Ritz, Poincaré, etc. The causes and implications of the failure of the Rayleigh–Jeans (R–J) formula though interesting, have not been so widely commented upon. Lord Kelvin[1] in his speech delivered on 27 April 1900 appraised the Royal Institution: 'The beauty and clearness of the dynamical theory, which asserts heat and light to be modes of motion, is at present obscured by two clouds.' This was a reference to (a) the null result of Michelson Morley experiment and (b) the failure of the law of equipartition of energy to explain the black body spectrum, i.e. the R–J formula.

The successful formula of Planck was based on a totally new idea, the energy quantization which had no classical basis. The R–J formula though unsuccessful had a very sound classical foundation. This strange situation induced a new wave of thinking in the minds of the physicists of the time who sought to look for the cause of this unexpected failure. First, the views of different luminaries of the time will be given and then the attempt of Einstein to clear the haze will be described in detail. Einstein pointed out that the cause of failure of R–J formula was not the inapplicability of the equipartition law but was more deep-rooted. The statistical fluctuations in radiation momentum are, in long and short wavelength regions, different from each other. The R–J formula was derived for long wavelengths compared with the dimensions of the Planck oscillators and the molecules. The calculation fails for shorter waves.

Attempts prior to Einstein's

Lord Rayleigh[2] was the first to admit the inapplicability of the equipartition law to the structureless aether and continuous radiation because the original formulation by Boltzmann aimed at its application to discrete systems, matter consisting of atoms and molecules. Although in the limit of long wavelengths the Planck-formula transforms into the R–J, Lord Rayleigh remarked, '... how another process also based upon Boltzmann's ideas can lead to a different result?' The R–J formula tells us that at a given temperature the total energy of radiation emitted by the black body is infinite. Lord Rayleigh attributed this to be the result of applying the equipartition law to a continuous medium which has infinite degrees of freedom and thus infinite specific heat. An emitter of radiation placed in such a medium will go on delivering energy to the medium till it loses all its energy and attains a state of 'absolute privation of heat' (Rankine) or absolute zero of temperature.

Lord Rayleigh commented on an earlier proposal of Jeans who had classified the molecular motions into two groups, translatory and vibratory. In the former class the equipartition could be established within a fraction of a second but in the latter it would take a million years or so. Lord Rayleigh remarked,

> '. . . does the postulated slowness of transformation really obtain? Red light falling upon the blackened face of a thermopile is absorbed and the instrument rapidly indicates a rise of temperature. Vibrational energy is readily converted into translational energy. Why then, does the thermopile not itself shine in dark?
>
> It seems to me that we must admit the failure of the law of equipartition in these extreme cases. If this is so it is obviously of great importance to ascertain the reason.'

In a later note[3], 'In the application to waves that are not long, there must be some limitation on the principle of equipartition.'

More restless was Jeans. He commented[4] first on the procedure of Planck[5]. He objected to the definition of entropy of a single resonator. However, he turned his attention to another aspect.

'Planck (in common, I know, with many other physicists) speaks of the "probability" of an event, without specifying the basis according to which the probability is measured. This conception of probability seems to me an inexact conception, and as such to have no place in mathematical analysis. For instance, a mathematician has no right, *qûa* mathematician, to speak of the probability that a tree shall be between six and seven feet in height unless he at the same time specifies from what trees the tree in question is to be selected and how... Prof. Planck's position is as though he attempted to calculate the probability that a tree should be between six and seven feet high, taking as his basis of calculation an enclosure of growing trees and assuming the probability to be a function only of the quantities six and seven feet...'

If one goes through the original paper of Planck[5] the above raised objections of Jeans appear too artificial. Planck started from the thermodynamic not the statistical definition of entropy which is $\mathrm{d}S/\mathrm{d}U = 1/T$, the value of $1/T$ was extracted out from Wien's formula* which on integration gave

$$S = -(U/a\nu)\log(U/eb\nu), \tag{1}$$

and yielded,

$$\mathrm{d}^2S/\mathrm{d}U^2 = \text{const.}/U. \tag{2}$$

On the other hand the temperature dependence of the black body radiation led to

$$\mathrm{d}^2S/\mathrm{d}U^2 = \text{const.}/U^2 \tag{3}$$

Planck's obvious compromise was

$$\mathrm{d}^2S/\mathrm{d}U^2 = \text{const.}/U(U+a). \tag{4}$$

Integration of this equation led to the first empirical formula that was a good fit for data covering the range from UV to IR. Thus far no use of statistical mechanics was made. Boltzmann's thermodynamic probability was applied to an assembly of oscillators, not to a single oscillator, which led to energy quantization. Jeans objected,

'Here ε is a small quantity, sort of indivisible atom of energy, introduced to simplify calculation. We may legitimately *remove this artificial quantity by passing to the limit in which* $\varepsilon = 0$.'

*Energy density of radiation in the frequency range, ν, $\nu + \mathrm{d}\nu$ is $U = (b\nu^3/c^2)\exp(-a\nu/T)$, where c is the velocity of light, a and b are constants

However Planck's logic was different. If energy U is to be distributed among N oscillators and if the smallest unit is zero there would be infinite ways of distributing U, making Boltzmann probability also infinite. Thus to keep it finite it was necessary to keep the smallest unit ε as non-zero. The theoretical basis was still lacking as was pointed out by Einstein as well[6].

Three years later there appeared a series of papers on the radiation problem in the journal *Physikalische Zeitschrift*. We find Jeans defending his formula. He compared his distribution function

$$8\pi RT\lambda^{-4}\,\mathrm{d}\lambda \tag{5}$$

with that of Planck,

$$8\pi RT\lambda^{-4}\,u\,\mathrm{d}\lambda/(e^u-1), \tag{6}$$

where $u = c/RT\lambda$, λ is the wavelength of radiation, R the gas constant and T the absolute temperature. Planck's formula predicted a maximum while that due to Jeans did not. Jeans[7] reasoning runs as follows. Jeans considers an acoustical analogy of thermal excitations in a black body. He considers a cavity filled with air (in place of aether) and fitted with a large number of ringing bells (instead of Planckian oscillators). Given sufficient time the entire sound energy will be transformed into molecular kinetic energy. This transformation, Jeans asserted, is consistent with his theory and not with Planck's. Jeans' expression can be Fourier-analysed showing the generation of infinitesimally small wavelengths. If sufficient time is not given these latter will not attain equilibrium with the cavity walls and one will indeed observe a maximum of intensity predicted by the Planck-formula. Thus Jeans claimed, there was nothing wrong with his formulas but in fact no experiment could ever be done to verify it. The observed maximum in the intensity distribution of the black body spectrum comes from short wavelength radiation that has not attained thermal equilibrium with the cavity. Jeans further asserted that his formula was the only one that was based on the dynamical principle of least action.

The present author however has a comment on this statement. The principle of least action in mechanics and the principle of entropy increase in thermodynamics were shown to be analogous to each other by Boltzmann[8]. If Jeans' formula was based on least action, Planck's was on entropy increase in the irreversible emission of radiation. Both had equally sound foundations. In later editions of his book Jeans[9] admitted that the failure of his law was due to the use of classical equations that were not valid in the quantum range.

Another person to derive a radiation formula on classical basis was Lorentz[10] who used Maxwell's electrodynamics, his own electron theory, kinetic theory of gases and the equipartition law. This work in all its detail was presented in a conference in Rome[11]. The ultimate result was again the R–J formula. The cavity of Lorentz was a parallelopiped made out of perfectly reflecting walls of a metal containing a large number of free electrons. Lorentz assumed in his calculations that the free electrons undergo slow transitions while the resonators undergo fast transitions. In the short wavelength region only resonators were active. If the centres of emission are electrons in electrodynamics, where do the electrons go when the short waves are emitted? No solution was found and Lorentz suggested that unless some drastic changes are made in the electron-theory, classical considerations can never lead to the correct radiation formula like that of Planck which was the only one acceptable expression at that time.

Ritz[12] was very critical of the use of retarded and advanced potentials by Lorentz. The retarded potential represents the outgoing while the advanced the incoming wave from infinity. The latter implies a continuous supply of energy from aether to the source without any effort, i.e. out of nothing. This contradicts the impossibility of perpetuum mobile and the second law of thermodynamics. When Einstein[13] pointed out that the smallness in the volume of cavity prevents the distinction between the retarded and advanced fields, Ritz[14] drew attention to a procedure adopted by Lorentz; association of surface integrals with these fields. The one associated with the retarded field gets vanished while the other linked with the advanced field remains non-zero. A joint note by Ritz and Einstein[15] was published declaring their differences of opinion. Ritz believed that the irreversibility of the emission process stemmed from the retarded potential while Einstein saw its origin in the probabilistic nature of emission (the present author thinks that this was a hint to the quantum view of emission).

How Einstein finally reasoned the cause of the failure of R–J formula

Einstein[16] considered a cavity full of radiation containing some gas molecules and resonators, the radiation interacted with the resonators and resonators with the colliding molecules. The equipartition law was applied to the molecules of the gas for which it was well tested. The motion of the resonator is subject to equal radiation pressure from all sides. This is how radiation damps the movement of the resonator. This loss in momentum of the resonator is compensated by the gain due to the statistical fluctuations in momentum of the radiation. This compensation resulted in a differential equation whose solution again turned out to be the R–J formula. Thus the equipartition law was not at fault. The momentum fluctuations for short and long wavelengths are different. A classical calculation is possible only for long waves where the constancy of the electric vector can be assumed over large distances and thus the field acting on different parts of a resonator is the same. The R–J formula is valid only for long waves satisfying this condition. As soon as the wavelength is comparable to the molecular or resonator dimensions the electric field of the radiation wave acting on the different parts of the resonator will not be the same, no classical averaging can yield the desired result.

Outline of the method

In order that a resonator can interact with a molecule via collision, a small inertial mass was associated with the resonator, this is m. The value of $(mv)^2$ can be calculated statistically – but there is also present a radiation field. The statistical equilibrium is possible only if the field does not modify the value of $\overline{(mv)^2}$ gained by resonator as a result of molecular collisions. Thus the moving oscillator interacting with the field also has a value $\overline{(mv)^2}$. How this happens is explained below.

In this context two kinds of forces are considered. First, the radiation damping. Cavity radiation in thermal equilibrium exerts at a given point inside the cavity equal pressure from all sides by virtue of detailed balancing to be explained after equation (31). It therefore opposes any translatory motion of the resonator (or oscillator). This force is proportional to the instantaneous velocity of the resonator and is called the damping force, denoted by $K = -Pv$. If this force acts for time τ, the momentum given to the resonator is $-Pv\tau$. The negative sign denotes loss in momentum due to damping. The second force comes from the radiation field fluctuations. The momentum gained by the resonator due to momentum fluctuations of the field was denoted by Δ. Let at time $t = 0$ the average of the square of the momentum gained by the resonator from molecular collisions be $\overline{(mv)^2_{t=0}}$. After time τ the loss in momentum is $Pv\tau$ and gain is Δ. Einstein equated,

$$\overline{(mv)^2_{t=0}} = \overline{((mv)_{t=0} + \Delta - Pv\tau)^2}. \quad (7)$$

The loss in damping was compensated by fluctuations. Neglecting τ^2 and $v \cdot \Delta$,

$$\overline{\Delta^2} = 2mv^2 P\tau. \quad (8)$$

The next step was to apply the equipartition law by replacing mv^2 by $(R/N) \cdot \Theta$, where Θ is the absolute temperature, R the gas constant and N the Loschmidt number. Equation (8) becomes,

$$\overline{\Delta^2} = 2(R/N)P \cdot \Theta \cdot \tau. \quad (9)$$

This equation was mentioned by Einstein in his Salzburg lecture[13] in 1909. Max Planck expressed his dissatisfaction on introducing gas molecules in the cavity. The

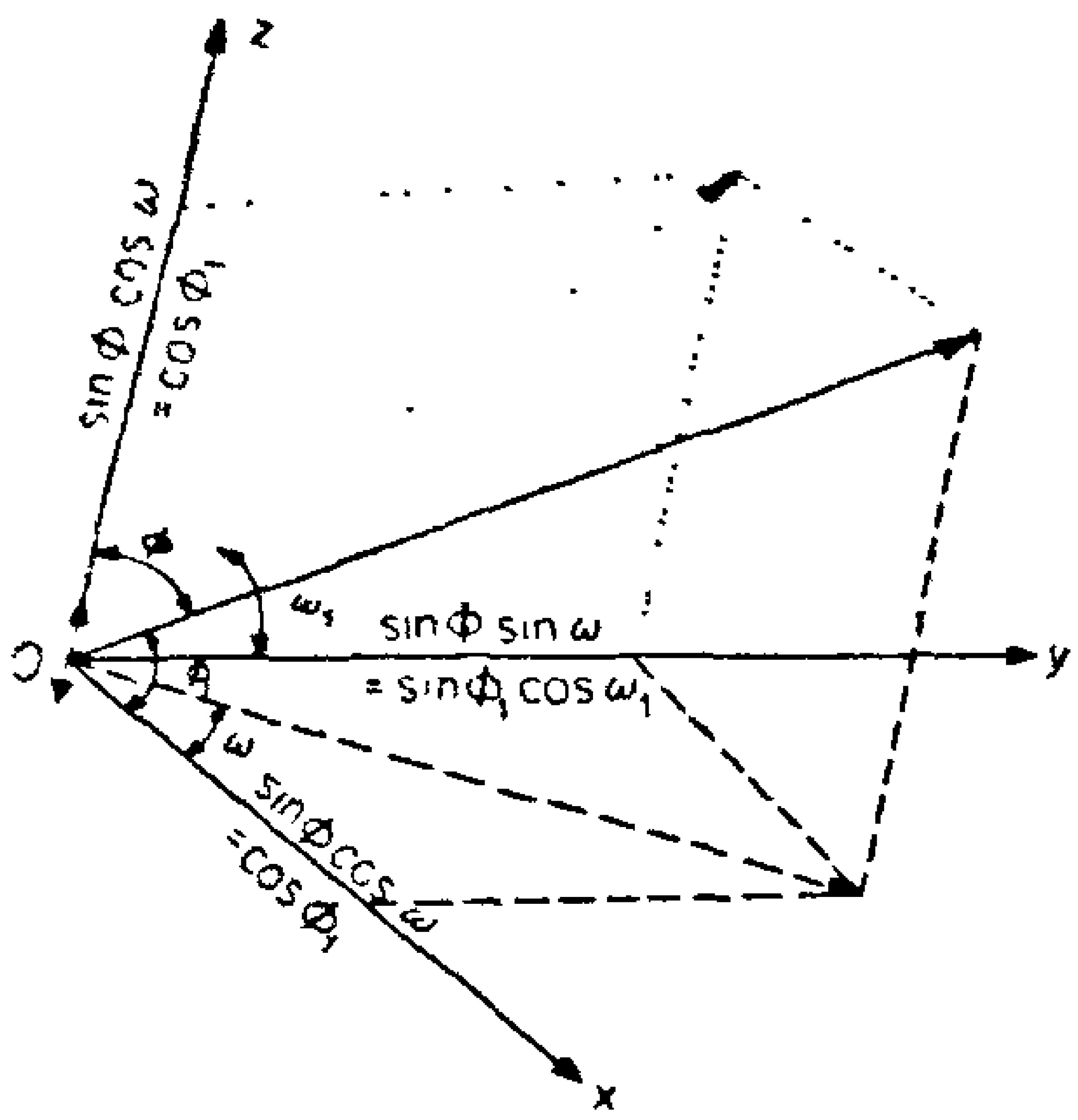

Figure 1. Components of a vector in terms of (φ, ω) and (φ_1, ω_1) along x, y, z axes. At 0 is shown the dipole oscillator

details of the quantum interaction of radiation and matter were then not well understood.

Calculation of the damping force on the resonator

Consider a resonator at rest at the origin of coordinates in Figure 1. It is oriented to vibrate along the z-axis. A ray is shown to be incident on it at an angle φ such that its projection on the xy-plane makes an angle ω on the x-axis. At this stage Einstein used Planck's mathematical apparatus developed for an electromagnetic resonator vibrating in the presence of an electromagnetic field. If f is the dipole moment of the resonator, ν_0 its characteristic frequency, $\mathcal{E}_z$ the electric vector of the incident field, σ, the damping constant and c the speed of light, Planck's equation of motion was[17]

$$16\pi^4 \nu_0^3 f + 4\pi^2 \nu_0 \ddot{f} - 2\sigma \dddot{f} = 3\sigma c^3 \mathcal{E}_z. \quad (10)$$

The electric field of the incident wave is expressed as

$$\mathcal{E} = \sum_n A_n \cos\left\{\frac{2\pi n}{T}\left(t - \frac{\alpha x + \beta y + \gamma z}{c}\right) - \theta_n\right\}, \quad (11)$$

where $\alpha = \sin\varphi \cos\omega$, $\beta = \sin\varphi \sin\omega$, $\gamma = \cos\varphi$ are the direction cosines, T is the time period of the wave and is very large compared to that of the resonator. The relevant field components required are $\mathcal{E}_x = \mathcal{E}\cos\varphi \cos\omega$, $\mathcal{E}_z = -\mathcal{E}\sin\varphi$ and $\mathcal{H}_y = \mathcal{E}\cos\varphi \sin\omega$. The force on the dipole is

$$\bar{k} = f \cdot \frac{d\mathcal{E}}{dz} + \frac{1}{c}\left[\frac{df}{dt}\mathcal{H}\right] \quad (12)$$

with the x-component

$$k_x = f \cdot \frac{\partial \mathcal{E}_x}{\partial z} - \frac{1}{c}\mathcal{H}_y \frac{df}{dt} \quad (13)$$

Planck expressed his solution to equation (10) as

$$f(t) = \frac{3c^3}{16\pi^3\nu^3} C \sin\gamma \cos(2\pi \nu t - \theta - \gamma), \quad (14)$$

which was first transformed by substituting in the value for $\mathcal{E}_z$. Using $\mathcal{E}_x$ in the expression for $\mathcal{E}$ and $\gamma = z\cos\varphi$ one obtains after a little calculation (which is being avoided here to reduce the article's length),

$$\frac{\partial \mathcal{E}_x}{\partial z} \cdot f = -\frac{3}{8}\frac{c^2}{\pi^2} T^2 (1 - \sin^2\varphi)\cos\omega \sin\varphi \times \sum_n \sum_m A_n \frac{\sin\gamma_n}{n^3} A_m m \cos(\tau_n - \gamma_n) \sin\tau_m \quad (15)$$

where $\tau_n = (2\pi n t/T - \theta_n)$

$$\frac{1}{c}\mathcal{H}_y \frac{df}{dt} = \frac{3}{8}\frac{c^2}{\pi^2} T^2 \sin\varphi \cos\varphi \times \sum_n \sum_m A_n \frac{\sin\gamma_n}{n^2} \sin(\tau_n - \gamma_n) A_m \cos\tau_m \quad (16)$$

Einstein used Planck's values,

$$\left.\begin{aligned} A_n &= \sum_0^\infty C_n^2 \frac{\sin^2\gamma_n}{\nu^2} \Big/ \overline{TC_n^2} \\ \frac{\sin^2\gamma_n}{\nu^2} &= T\langle A\rangle = T\frac{\sigma}{2\nu_0} \end{aligned}\right\} \quad (17)$$

and arrived at the average damping force on a stationary resonator,

$$\overline{k_x} = \frac{3}{16}\frac{c^2}{\pi^2} T \sin^3\varphi \cos\omega\, \overline{A^2_{\nu_0 T}}\, \frac{\sigma}{2\nu_0}. \quad (18)$$

Let the oscillator now move along the axis of x with a velocity v. The above expression will be transformed using Lorentz transformations. The calculation is eased by introducing new variables φ_1, ω_1 as shown in Figure 1 by dashed lines. These new angles are related to the older ones by $\sin\varphi_1 \cos\omega_1 = \sin\varphi \sin\omega$, $\sin\varphi_1 \sin\omega = \cos\varphi$ and $\cos\varphi_1 = \sin\varphi\cos\omega$. The Lorentz transformations used were

$$A' = A[1 - (v/c)\cos\varphi_1] \quad (19a)$$

$$T' = T[1 + (v/c)\cos\varphi_1] \quad (19b)$$

$$v' = v\,[1 - (v/c)\cos\varphi_1] \tag{19c}$$

$$\tan\varphi_1' = [\sin\varphi_1(1 - (v^2/c^2)]/(\cos\varphi_1 - v/c) \tag{19d}$$

$$\omega_1' = \omega. \tag{19e}$$

It is to be remarked that no mass transformation was involved! Planck's resonators were massless, only electromagnetic interactions were considered, the inertial ones were not taken into account in order to keep the black body spectrum independent of material.

Equations (19) led to

$$\cos\varphi_1' = [\cos\varphi_1 - (v/c)]/[1-(v/c)\cos\varphi_1] \tag{20}$$

$$\overline{A'^2_{v_0'T'}} = \{A^2_{v_0'T'} + v_0'\,(v/c)\cos\varphi_1\,(d\overline{A_1^2}/dv)_{v_0T}\}$$

$$\times\,[1 - 2(v/c)\cos\varphi_1]. \tag{21}$$

Using $\sin^3\varphi\cos\omega = (1-\sin^2\varphi_1'\sin\omega_1')\quad\cos\varphi_1$, the expression for the average of the x-component of the damping force turned out to be,

$$\overline{k_x'} = (3c^3/16\pi^2)\{A^2_{v_0T} + v_0'(v/c)\cos\varphi_1(dA^2/dv)_{v_0'T}\}$$

$$\times\,[1-(2v/c)\cos\varphi_1]\,T[1-(v/c)\cos\varphi_1](\sigma/2v_0^1)$$

$$\times\,(1-\sin^2\varphi_1'\,\sin^2\omega_1')\cos\varphi_1'. \tag{22}$$

This was transformed to include the radiation density ρ and the solid angle $d\kappa$ by

$$\left.\begin{aligned}\rho(d\kappa/4\pi) &= \frac{1}{8\pi}\cdot\frac{\overline{A^2T}}{2}(2)\cdot(2)\\ \overline{A^2} &= \rho\,d\kappa/T.\end{aligned}\right\} \tag{23}$$

Neglecting the terms of $(v/c)^2$ and putting $\omega_1' = \omega$ equation (22) is,

$$\overline{k_\kappa'} = \left(\frac{3}{16}\right)(c^2/\pi^2)(\sigma/2v_0')\{\rho_{v_0'} + v_0'\,(v/c)$$

$$\times\cos\varphi_1(d\rho/dv)_{v_0'}\}\;(1-\sin^2\varphi_0'\,\sin\omega_1)$$

$$\times\,[\cos\varphi_1 - (v/c)]\,d\kappa. \tag{24}$$

With $d\kappa = \sin\varphi_1\,d\omega_1\,d\varphi_1$ and integrating in the limits 0 to 2π for ω_1, 0 to $\pi/2$ for φ_1 one gets

$$K = -(3c/10\pi)(\sigma/2v_0')\,v\,\{\rho + (v/c)\rho$$

$$-\,v_0'(v/c)\,1/3\,(d\rho/dv)_{v_0'}\}. \tag{25}$$

The constant term ρ is independent of v, therefore the damping force by virtue of the motion of the resonator is,

$$K = (3c^2/10\pi)(\sigma/v_0')(v/c)\left\{\rho_{v_0'} - \frac{v_0'}{3}\left(\frac{d\rho}{dv}\right)_{v_0'}\right\} \tag{26}$$

Calculation of $\overline{\Delta^2}$

Let the impulse received by the resonator in time τ from radiation be J. Equation (13) leads to

$$J = \int_0^\tau k_z\,dt = \int_0^\tau\left(\frac{d\mathcal{E}_x}{dz}f - \frac{1}{c}\mathcal{H}_y\frac{df}{dt}\right)dt, \tag{27}$$

$$\frac{1}{c}\int_0^\tau \mathcal{H}_y\frac{df}{dt}\,dt = \frac{1}{c}[\mathcal{H}_y\,f]_0^\tau - \frac{1}{c}\int_0^\tau\frac{\partial\mathcal{H}_y}{\partial t}f\,dt. \tag{28}$$

For τ large the first term is zero in (28) because in a vibratory motion the displacement is as much positive as is the negative. The Maxwell equation

$$\frac{1}{c}\frac{\partial\mathcal{H}_y}{\partial t} = \frac{\partial\mathcal{E}_z}{\partial x} - \frac{\partial\mathcal{E}_x}{\partial z} \tag{29}$$

gives for the second term in (28),

$$\frac{1}{c}\int_0^\tau\mathcal{H}_y\,\frac{df}{dt}\,dt = -\int_0^\tau\left\{\left(\frac{\partial\mathcal{E}_z}{\partial x}\right)f - \left(\frac{\partial\mathcal{E}_x}{\partial x}\right)f\right\}dt \tag{30}$$

and

$$J = \int_0^\tau\frac{\partial\mathcal{E}_z}{\partial x}f\,dt. \tag{31}$$

At this stage the principle of detailed balancing is used. Thermal equilibrium in a cavity implies the stationary state in which there is no net flow of energy in any direction. To every ray with a given amplitude, frequency and polarization there is a corresponding ray with the same characteristics travelling in the opposite direction. Thus,

$$\mathcal{E}_z = \sum\left\{a_n\sin\frac{2\pi n}{T}\left(t - \frac{\alpha x+\beta y+\gamma z}{c}\right)\right.$$

$$+\,b_n\cos\frac{2\pi n}{T}\left(t - \frac{\alpha x+\beta y+\gamma z}{c}\right)$$

$$+\,a_n'\sin\frac{2\pi n}{T}\left(t + \frac{\alpha x+\beta y+\gamma z}{c}\right)$$

$$+h_n' \cos\frac{2\pi n}{T}\left(t+\frac{\alpha x+\beta y+\gamma z}{c}\right)\Big\} \quad (32)$$

and

$$\mathcal{E}_z = \sum_n B_n \cos\left(\frac{2\pi nt}{T}-\theta_n\right),$$

$$\frac{\partial \mathcal{E}_z}{\partial x} = \sum C_m \cos\left(\frac{2\pi nt}{T}-\xi_m\right). \quad (33)$$

These equations and the solution (14) when substituted in (31) give,

$$J=\frac{1}{2}\frac{3c^3}{8\pi^3}T^3\int_0^\tau dt \sum_m\sum_n C_m B_n \frac{\sin\gamma_n}{n^3}$$

$$\times\left[\cos\left\{2\pi\ (n+m)\frac{t}{T}-\xi_m-\theta_n-\gamma_n\right\}\right.$$

$$\left.-\cos\left\{2\pi\ (n-m)\frac{t}{T}+\xi_m-\theta_n-\gamma_n\right\}\right] \quad (34)$$

On integration two factors, $(m+n)^{-1}$ and $(m-n)^{-1}$ appear, the former is too small for large m and n and is neglected. We are left with

$$J=-\frac{3c^3}{32\pi^4}T^4\sum_m\sum_n C_m B_n\frac{\sin\gamma_n}{n^3}$$

$$\times\frac{1}{(n-m)}\cos\delta_{mn}\sin\pi\ (n-m)\frac{\tau}{T} \quad (35)$$

$$\delta_{mn}=\pi\ (n-m)\frac{\tau}{T}+\xi_m-\theta_n-\gamma_n. \quad (36)$$

The square of J is a four-fold sum over $m-n$ and two new indices m', n', δ_{mn}, $\delta_{m'n'}$ are angles independent of each other. This independence is preserved for $m=m'$ and $n=n'$. Summing over n implies $t\to\tau$. Putting $n=\nu T$ and $m=\mu T$ and replacing Σ_m by $\int dm$ the term inside the square brackets is

$$(1/T)\int_0^\infty\frac{1}{(\nu-\mu)^2}\sin^2\pi(\nu-\mu)\tau\, d\mu=n^2\tau/T. \quad (37)$$

Since,

$$\sum_n(\sin^2\gamma_n/n^6)=(1/T^5)\cdot(\sigma/2\nu_0^5) \quad (38)$$

$$\overline{J^2}=(3c^3/32\pi^3)^2\ (\sigma\ \tau/4\nu_0^5)\overline{C^2_{\nu_0 T}}\ \overline{B^2_{\nu_0 T}}\cdot T^2 \quad (39)$$

Now,

$$\overline{J^2}=\overline{(\bar{J}+\Delta)^2}=\bar{J}^2+2\bar{J}\cdot\Delta+\Delta^2 \quad (40)$$

$$\bar{J}=\bar{\Delta}=0 \quad (41)$$

$$\overline{J^2}=\overline{\Delta^2} \quad (42)$$

Evaluation of $\overline{B^2_{\nu_0 T}}$ and $\overline{C^2_{\nu_0 T}}$ remains. The former is $\sin\varphi$ component of $A_{\nu_0 T}$,

$$(B_{\nu_0 T})^2=\left(\sum A_{\nu_0 T}\sin\varphi\right)^2$$

$$B^2_{\nu_0 T}=\overline{A^2_{\nu_0 T}}\sum\sin^2\varphi \quad (43)$$

Using (23)

$$B^2_{\nu_0 T}\cdot T=\int\sin^2\varphi\ \cdot\rho_{\nu_0}d\kappa=\frac{8}{3}\pi\ \rho_{\nu_0} \quad (44)$$

By equations (33), $C_{\nu_0 T}$ is the projection of the derivative of $B_{\nu_0 T}$ along the x-axis,

$$\overline{C^2_{\nu_0 T}}\cdot T=\left(\frac{2\pi\nu}{c}\right)^2\overline{A^2_{\nu_0 T}}\sum\sin^4\varphi\cos^2\omega$$

$$=\frac{64}{15}\frac{\pi^3\nu_0^2}{c^2}\rho_{\nu_0}. \quad (45)$$

Substituting for $B_{\nu_0 T}$ and $C_{\nu_0 T}$ in (39) and using (41),

$$\overline{\Delta^2}=(c^4\sigma\ \tau/40\pi^2\nu_0^3)\rho^2_{\nu_0}. \quad (46)$$

Equations (26), (46) and (9) yield the differential equation in ρ,

$$(c^3N/24\pi R\Theta\nu^2)\rho^2=\rho-\frac{\nu}{3}\frac{d\rho}{d\nu} \quad (47)$$

whose solution is the R–J formula

$$\rho=8\pi R\ \Theta\nu^2/c^3N \quad (48)$$

resulting from the application of the equipartition law to the molecules, not to aether or radiation.

Einstein's conclusions

The above calculation showed that the law of equipartition had nothing to do with the failure of the

R–J formula. To understand the role of $\overline{\Delta^2}$, the famous thought experiment[13] presented by Einstein at the Salzburg lecture is mentioned below.

The resonator in the cavity is replaced by a freely moving front-silvered mirror plate having a mass of the order of molecular mass. Since the radiation is reflected only in the front surface, there will be a difference in radiation pressure on the two sides, giving rise to the damping force analogous to that given by equation (26) above,

$$P = (3/2c)\,[\,\rho - (1/3)\,(\nu)\,(\mathrm{d}\rho/\mathrm{d}\nu)]\mathrm{d}\nu\cdot f, \qquad (49)$$

where f is the mirror surface area. A further supposition is made. The mirror reflects selectively only in the range ν, $\nu+\mathrm{d}\nu$, to all other frequencies it is transparent. Equation (9) then reads[10]

$$\frac{\overline{\Delta^2}}{\tau} = \frac{RA}{N}\,\frac{3}{c}\left[\rho - \frac{\nu}{3}\,\frac{\mathrm{d}\rho}{\mathrm{d}\nu}\right]\mathrm{d}\nu\cdot f. \qquad (50)$$

The expression for ρ can be substituted either from Wien's, or R–J's or from Planck's radiation formulae. Planck's formula gives,

$$\frac{\overline{\Delta^2}}{\tau} = \frac{1}{c}\left[h\rho\nu + \frac{c^3}{8\pi}\,\frac{\rho^2}{\nu^2}\right]\mathrm{d}\nu\cdot f. \qquad (51)$$

The first term represents the quantum while the second the wave nature of radiation i.e., the dual nature of radiation came out for the first time at the Salzburg lecture. If Wien's law is used only the first term comes out, while the R–J formula gives only the second. The nature of $\overline{\Delta^2}$ for long and short wavelengths is different. R–J formula was derived for long wavelengths only and therefore it failed for the short ones. This was Einstein's conclusion.

1 Lord Kelvin, *Philos Mag*, 1901, II, 1
2 Lord Rayleigh, *Nature*, 1905, **72**, 54
3 Lord Rayleigh, *Nature*, 1905, **72**, 243
4 Jeans, J H, *Nature*, 1905, **72**, 293
5 Planck, M, *Ann Phys*, 1900, **1**, 69 and *Naturwissenschaften*, 1943, **31**, 153
6 Einstein, A, *Ann Phys*, 1906, **20**, 199
7 Jeans, J H, *Phys Zeit*, 1908, **9**, 853
8 Boltzmann, L, *Wiener Berichte*, 1866, **53**, 195
9 Jeans, J H, *Dynamical Theory of Gases*, 1925, 4th edition reprinted by Dover, London, 1954
10 Lorentz, H A, *Phys Zeit*, 1908, **9**, 562
11 Lorentz, H A., Sharing of energy between ponderable matter and aether, conference tenue au congrés de Rome, Rome Tipografia della R Accad, de Lincei, avril, 1908
12 Ritz, W, *Phys Zeit*, 1908, **9**, 903
13 Einstein, A, *Phys Zeit*, 1909, **10**, 185, 817.
14. Ritz, W, *Phys Zeit*, 1909, **10**, 224
15 Ritz, W and Einstein, A, *Phys Zeit*, 1909, **10**, 323
16 Einstein, A and Höpf, L, *Ann Phys*, 1910, **33**, 1105
17 Planck, M, *Vorlesungen uber die Theorie der Warmestrahlung*, p 122, Verlag von Iohann Ambrosius Barth, Leipzig, 1906 The later editions of this book give a different treatment
18 Klein, M J, Article No 12, 'No Firm Foundation Einstein and the early quantum theory' in *Some Strangeness in Proportion* (ed Harry Wolf), A centennial symposium to celebrate the achievements of Albert Einstein, Addison Wesley Advanced Book Program, 1980 This gives an English version of Einstein's thought experiment.

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Amar Nath Nigam (formerly at HBTI Kanpur) lives at 8/20 Arya Nagar, Kanpur 209 002, India

Erratum

The article 'Leaf gas exchange in lightflecks of plants of different successional range in the understorey of a Central European beech forest' by Kailash Paliwal *et al.* was published under the category of 'Review Article'. However, it should have been published under the 'Research Article' category. We regret the error.

– *Editor*

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