Incomparability of systems with multiple equivalent quantitative descriptions

P. S. Moharir

The Miller paradox, that the ranking of systems may depend on which among the equivalent quantitative descriptions is used for comparison, is ubiquitous and readily constructible. It is extended here to induced Condorcet paradox. The original Condorcet paradox implies that multi-criterion problems may not have rational solutions. The variety induced under equivalent descriptions implies that the systems which support multiple alternative equivalent quantitative descriptions are not properly comparable.

PARADOXES are fun. But often they bring in their wake fundamental problems. Occasionally, these problems can be solved leading to phenomenal progress, as happened in the case of Galileo's paradox. When such a development does not take place, paradoxes bring out limitations on our power to infer. Russell's paradox[1] led to a clearer understanding of the limitations of axiomatic thinking. This paper deals with Miller paradox[2,3], which too is believed to have equally far-reaching implications.

Does the barber in a certain village, who shaves all and only those in the village, who do not shave themselves, shave himself, Russell[1] asked. The crocodile who had stolen a child promised the parent to return the child, provided the parent correctly guesses whether it will return the child or not. If the parent guesses that the crocodile will not return the child, what should it do[4]? Such stories do a round in the post-dinner session and are not normally allowed to do more than tantalize us for a brief while. Some responses are intelligent, but only at the level of the popularized version of the paradoxes. For example, one may deflate the air in Russell's paradox by saying that 'himself' in it is only a vestige of masculine bias in the language and the barber was a woman. Russell's question, in fact, is whether the set, of all sets, which are not members of themselves, is a member of itself[4]. Hence, the question of gender is totally irrelevant, and the paradox and its implications remain. The implications are immense. One of the most ambitious mathematical programmes is called Hilbert's programme[5,6] (or his Entscheidungsproblem[7]). The search was for a general mechanical procedure which will (at least, in principle) settle all the problems of mathematics. *Principia Mathematica*[8] was motivated by the urge to set up such an axiomatic system. Godel[9], working on Hilbert's programme, ended up proving that every axiomatic system, general enough to include the axioms of arithmetic, was either inconsistent or incomplete. That is, within such a system, it was possible to frame propositions, whose truth-status could not be determined. Similar conclusions were independently reached by Church[10] and Turing[11]. It is believed that[7] all these proofs underscoring the defeat of Hilbert's programme basically precipitate Russell's paradox intelligently. It is this that made Frege' say that[3] arithmetic has tottered due to Russell's paradox.

Is physics tottering, Popper[3] asked, after discussing Miller's paradox. He believed that the conclusion may be hasty and that the semantics of the problem should normally help in resolving the paradox, though he has not made any specific suggestions regarding how. Paradoxes can indeed lead to progress, as happened in the case of Galileo's paradox. Galileo was puzzled that there was 1 : 1 correspondence between the integers and perfect squares, though the set of latter constituted a proper subset of the former. Bolzano[12] showed that similar correspondences between the elements of an infinite set and a proper subset are almost ubiquitous. Historians[6] say: 'Dedekind saw in Bolzano's paradoxes not an anomaly, but a universal property of infinite sets which he took as a precise definition.' Dedekind's definition was a significant progress in mathematics.

The plan of this paper is to illustrate Miller paradox, to argue that it is ubiquitous and can be readily constructed, to extend it to induced Condorcet paradox and to suggest that all this implies an incomparability of systems which can be quantitatively described in many equivalent ways.

P. S. Moharir is in the National Geophysical Research Institute, Uppal Road, Hyderabad 500 007, India

Miller paradox

Consider that a matrix A transforms a vector $x = (x^{(1)}, x^{(2)}, \ldots)$ into a vector y as

$$y = Ax \tag{1}$$

such that the matrix A has an inverse. Then x and y are equivalent and interconvertible descriptions of some system. Let the true theory be that $x = x_0$ or equivalently that $y = y_0$. Miller[2] pointed out that given two alternatives I and II that $x = x_i, i = 1, 2$ or equivalently that $y = y_i, i = 1, 2$, it is possible that x_1 is closer to x_0 than is x_2 (I is better) and yet y_2 is closer to y_0 than is y_1 (II is better!). This has been called proximity reversal[13] under equivalent descriptions. The equivalence then is equivocal[14] when comparisons are to be performed.

As an example, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{2}$$

which is the Hadamard matrix[15] of order 2. Therefore, y is the Hadamard transform of x. Let

$$\Delta x_1 = x_1 - x_0 = (5, 11)',$$

$$\Delta x_2 = x_2 - x_0 = (7, 10)' \tag{3}$$

where overhead prime represents transposition. Then, Eq. (1) gives

$$\Delta y_1 = y_1 - y_0 = (16, 6)',$$

$$\Delta y_2 = y_2 - y_0 = (17, 3)'. \tag{4}$$

Let the proximity between vectors be measured by the L_1 norm of their differences. Then,

$$\| \Delta x_1 \|_1 = 16, \; \| \Delta x_2 \|_1 = 17 \tag{5}$$

so that x_1 is closer to x_0 than is x_2. However,

$$\| \Delta y_1 \|_1 = 22, \; \| \Delta y_2 \|_1 = 20 \tag{6}$$

so that y_2 is closer to y_0 than is y_1. Thus, by Hadamard transformation there has been proximity reversal. Miller[2] said that the results of metrical comparison among theories could change by transforming the theories into their logically equivalent forms. So Eqs (1), (2), (5) and (6) constitute an example of Miller paradox.

The example above was specifically chosen to illustrate other things. Hadamard transform satisfies Parseval's theorem. Therefore, there would be no proximity reversal if the measure of proximity is based on the L_2 norm

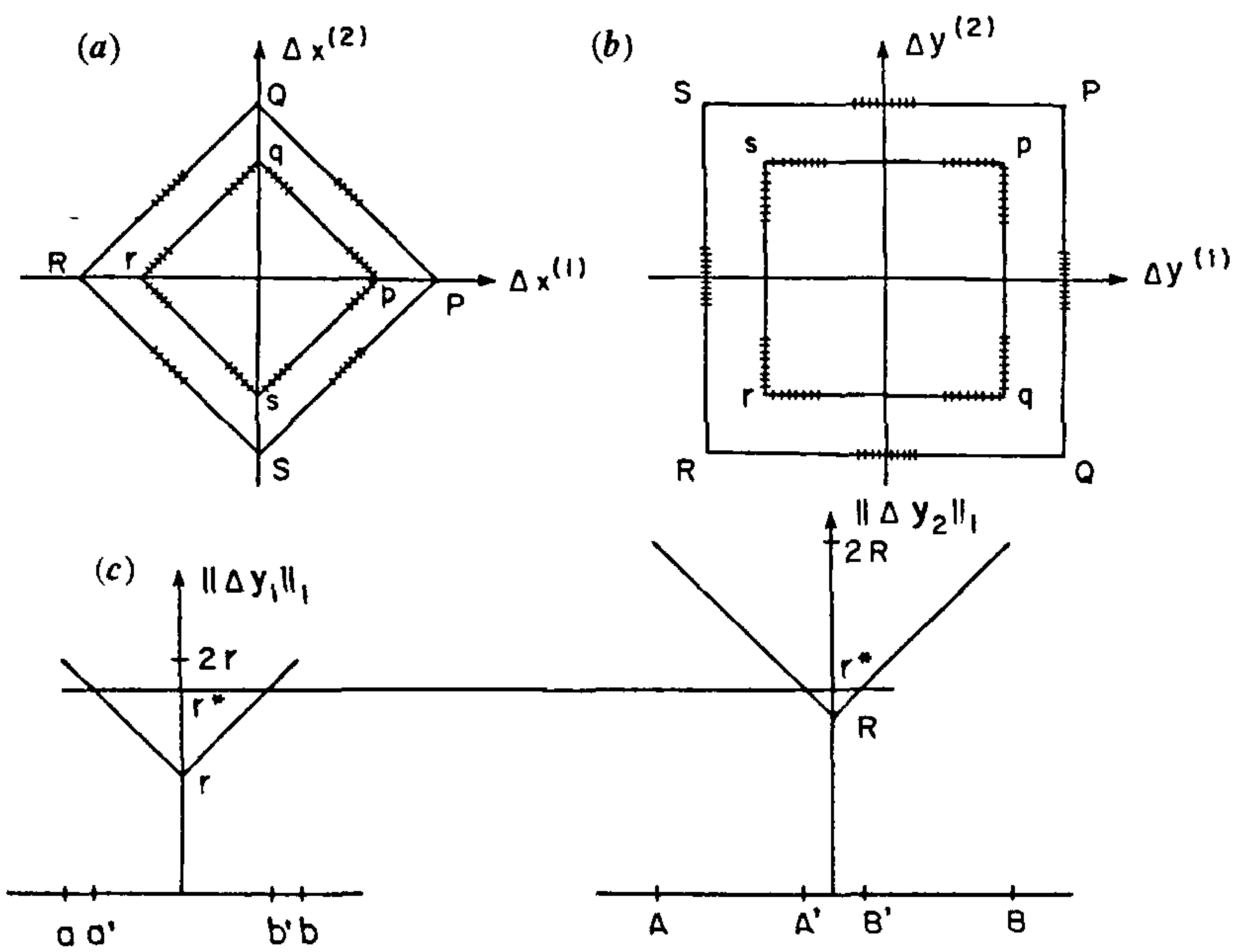

Figure 1. Study of Miller paradox under Hadamard transformation. ***a***, Equiproximate contours from x_0 are diamonds *pqrs* and *PQRS* if L_1 distance is used. ***b***, These diamonds become squares under Hadamard transformation. ***c***, Distances from y_0 for the two squares of (*b*).

of the differences. Further, if the proximity is measured in x and y domains by the L_1 and L_∞ norms of the differences, respectively, in the above example there is no proximity reversal. A question may be asked as follows: Let there be a norm which can engender proximity reversal and other paradoxes associated with it. Then does the use of another norm or a pair of norms for two equivalent descriptions which can avoid them, amount to staying away from the problem or solving it?

Next, it will be established that proximity reversals under equivalent transformations are quite ubiquitous.

Consider the same transformations as in Eqs. (1)–(2), choose

$$\|\Delta x_1\|_1 = r, \quad \|\Delta x_2\|_1 = R, \; r<R. \tag{7}$$

Then Δx_1 lies on the diamond pqrs and Δx_2 on the diamond PQRS, in Figure 1*a*. Under the Hadamard transformation, the two diamonds become squares *pqrs* and *PQRS* in the Δy domain in Figure 1*b*, but obviously the L_1 norms are not constant on the two squares (though L_∞ norms would be). The L_1 norms are plotted in Figure 1*c*, wherein the stretches ab and AB stand for any of the sides of the two squares respectively. That is, a is p, q, r or s and b then is q, r, s or p respectively, etc. Let

$$2r > r* > R. \tag{8}$$

Draw a horizontal straight line at an height $r*$ in Figure 1*c*. Then if Δy_1 is in the region marked aa' or $b'b$ and Δy_2 is in the region $A'B'$, that is if $\Delta x_i, i = 1,2$ are in the hatched regions in Figure 1*a* and $\Delta y_i, i = 1,2$ correspondingly in the hatched regions of Figure 1*b*, then

$$\|\Delta y_1\|_1 > \|\Delta y_2\|_1 . \tag{9}$$

Thus Eqs. (7) and (9) together imply a proximity reversal under Hadamard transformation and the choice of L_1 distance.

As an illustration that Miller paradoxes of larger order can be constructed from those of smaller orders, consider

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \; \boldsymbol{v} = \boldsymbol{B}\boldsymbol{u} \tag{10}$$

wherein $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular, invertible matrices. Let

$$\|\Delta x_1\|_1 < \|\Delta x_2\|_1 \text{ but } \|\Delta y_2\|_1 < \|\Delta y_1\|_1, \tag{11}$$

and

$$\|\Delta u_1\|_1 < \|\Delta u_2\|_1 \text{ but } \|\Delta v_2\|_1 < \|\Delta v_1\|_1, \tag{12}$$

so that both $\boldsymbol{A}$ and $\boldsymbol{B}$ engender proximity reversals with the use of a prescribed norm of differences to define proximity. Then, define

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B}, \quad \boldsymbol{w} = \boldsymbol{C}\boldsymbol{z} \tag{13}$$

where $\otimes$ denotes Kronecker product[16]. Then $\boldsymbol{C}$ engenders proximity reversal when

$$\Delta z_i = \Delta x_i \otimes \Delta u_i, \; i = 1,2 \tag{14}$$

so that

$$\|\Delta z_1\|_1 < \|\Delta z_2\|_1 \text{ but } \|\Delta w_2\|_1 < \|\Delta w_1\| \tag{15}$$

Further[17]

$$\Delta w_i = \Delta y_i \otimes \Delta v_i, \; i = 1,2. \tag{16}$$

The discussion above is adequate to illustrate that Miller paradox must be ubiquitous as it can be readily generated and enlarged to a system of larger order from systems of smaller orders. The equivalent transformations can be nonlinear as illustrated elsewhere[13] and could be drawn from real physical and mathematical contexts rather than being contrived.

Condorcet paradox and Arrow's impossibility theorem

There is a paradox called the voting paradox or the cyclic majority paradox, first supposed to have been discussed[18] by M. Condorcet. As a result, it is also called Condorcet paradox. Let there be three voters x, y and z and three candidates I, II and III. Let the voters express their preferences (> : 'is preferred to') as follows

$$\begin{aligned} x &: \text{I} > \text{II} > \text{III}, \\ y &: \text{II} > \text{III} > \text{I}, \\ z &: \text{III} > \text{I} > \text{II} \end{aligned} \tag{17}$$

There is nothing surprising that individual voters have different preference patterns. If the individual preferences are integrated into a collective preference pattern by a majority rule,

$$M: \text{I} > \text{II} > \text{III} > \text{I}, \tag{18}$$

where M stands for 'the majority'. Thus, the preference pattern of the majority is cyclic, which is not acceptable. One of the interesting reasons for inacceptability is economic. If you prefer a red ball to a blue one, and blue to green and green to red, I can give you a red ball instead of the blue you have for Re. 1/-, then green instead of red for Re. 1/-, and then blue instead of green for Re. 1/-. So, you have the blue ball you had and I have Rs. 3/- merely because you had a cyclic preference pattern. In fact, if you persist in your preference pattern, I can soon have Rs. 3*n*/-, *n* arbitrarily large!

To understand (18), it may be relevant to note that 'the majority *M*' stands for different pairs among *x*, *y* and *z* for different rankings in (18). For I > II, *M* is *x* and *z*; for II > III, it is *x* and *y* and for III > I, it is *y* and *z*. Further, while concurrence of *x* and *z* regarding I > II is accepted by *M*, their extreme disagreement about III is left out. Similar statements hold for other pairs of voters officiating as 'the majority' for other pairwise rankings. Thus the majority is poikilomorphic and integrates by parts.

Condorcet paradox led to Arrow's impossibility theorem[18], which states that there is no way of integrating individual preference patterns among more than three alternatives into a collective preference pattern, which is guaranteed to be noncyclic, provided certain reasonable conditions are insisted upon. He requires that (a) the preference ranking among a set of any three alternatives by any voter should not change if some more alternatives are added to that set, (b) the association between individual preferences and the collective preference ranking should not be negative, (c) the collective preference pattern should not depend on irrelevant alternatives, (d) it should not be imposed in that it should not be prescribed independent of what the individual voters say, and (e) it should not be dictatorial in that the preference ranking of one particular voter should not prevail as a collective preference ranking, overruling what other voters opine. Arrow's impossibility theorem is essentially a metaphor and implies that if there are more than three alternatives, a multi-criterion problem need not have a rational solution. Of course, for rigour and avoidance of over-simplification due to popularization, it is essential to refer to the original[18] definitions and proofs.

Induced Condorcet paradox

Consider matrices *A* and *B* which transform a vector *x* as follows

$$y = Bz, \quad z = Ax \tag{19}$$

such that matrices *A* and *B* are nonsingular. Then x, y, z are three equivalent and interconvertible descriptions of a system. Let the true theory be $x = x_0$ or equivalently that $y = y_0$, $z = z_0$. Let theories I, II and III be as defined earlier in the section entitled Miller paradox. As a particular example,

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 3 & 1 \\ 2 & 8 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -6 & 1 \\ -3 & 1 & 1 \\ 5 & 0 & 2 \end{bmatrix} \tag{20}$$

Then if

$$\Delta x_1 = (1, 2, -1)', \ \Delta x_2 = (-2, 3, 3)', \quad \Delta x_3 = (3, -4, 5)' \tag{21}$$

the transformations would give

$$\Delta y_1 = (-21, -4, 14)', \ \Delta y_2 = (1, 1, 0)', \quad \Delta y_3 = (5, -2, 2)' \tag{22}$$

and

$$\Delta z_1 = (8, 7, 13)', \ \Delta z_2 = (14, 8, 35)', \quad \Delta z_3 = (0, -1, 1)'. \tag{23}$$

Then, irrespective of which L_p norm of the difference is used to define proximity, we have:

$$\begin{aligned} x &: \text{I} > \text{II} > \text{III}, \\ y &: \text{II} > \text{III} > \text{I}, \\ z &: \text{III} > \text{I} > \text{II}, \end{aligned} \tag{24}$$

where the symbol on the left of the colon indicates the domain in which comparisons are made. The form of (24) is the same as that of (17) and yet the cyclic majority paradox that would result from (24) is more perplexing because now *x*, *y* and *z* are not three independent voters as in (17) but are three equivalent and inter convertible descriptions of a system. Hence the paradox is called 'induced Condorcet paradox'. Another example of it was given earlier[14]. Indeed, extending the procedures given earlier under the section Miller paradox, these paradoxes too can be readily constructed.

Now, the question raised earlier can be answered. Given a pair of equivalent descriptions, let there be a norm which engenders proximity reversal. Then to use another norm or a pair of norms which can avoid it would violate Arrow's postulate of the integration rule having to be nondictatorial.

Conclusion

If the implication of the Condorcet paradox is that multi-criterion problems may not have rational solutions, the connotation of the proximity reversal paradoxes is that systems amenable to multiple equivalent quantitative descriptions cannot be properly compared. A corollary is that if systems are studied with one such description and comparative conclusions drawn, study in terms of alternative, though equivalent, descriptions should not be neglected, because it can throw unsuspected new light. Equivalence of descriptions applies to individual systems and is not necessarily carried over to their inter-comparison.

Returning to Popper's questions whether with these paradoxes physics is tottering or not and whether the semantics of the problem would indicate which among the logically equivalent descriptions to trust for comparison, suffice it to say that these paradoxes indicate that mathematics does not take any semantic responsibility. It should not be assumed that mathematics only potentiates or facilitates. It can be a source of troubles too! Galileo's paradox eventually led to remarkable progress in mathematics. Proximity reversal paradoxes make a strong case for a newer understanding of the interaction between mathematics, a formal science, and empirical sciences which habitually rely on mathematics merely for aid, forgetting that it too can be problematic.

1. Russell, B, *Introduction to Mathematical Philosophy*, Allen & Unwin, London, 1919, pp 208
2. Miller, D. W, *The Accuracy of Predictions*, *Synthese*, 1975, **30**, 159–199
3. Popper, K. R., *Objective Knowledge, An Evolutionary Approach*, Clarendon Press, Oxford, 1979, pp 372–374.
4. Kleene, S C., *Introduction to Metamathematics*, North-Holland, Amsterdam, 1962, pp. 37–39
5. Hilbert, D, *Bull Am. Math Soc*, 1902, **8**, 437–439.
6. Boyer, C. B. and Merzbach, U C., *A History of Mathematics*, John Wiley, New York, 1989, pp 632.
7. Penrose, R., *The Emperor's New Mind Concerning Computers, Minds, and the Laws of Physics*, Oxford University Press, Oxford, 1989, pp 47–49, 57–64, 99–102.
8. Russell, B and Whitehead, A. A, *Principia Mathematica*, Cambridge University Press, Cambridge, 1962.
9. Godel, K., *Monat Math Phys*, 1931, **38**, 173–198.
10. Church, A., *Ann. Math Stud.*, 1941, **6**
11. Turing, A M, *Proc. London Math. Soc*, ser. 2, 1937, **4**, 230–265.
12. Bolzano, B, *Paradoxien des unendlichen*, 1850 (Translator Steele, D. A), *Paradoxes of the infinite*, Routledge & Kegan Paul, London, 1950
13. Moharir, P. S., *Induced Condorcet Paradox* (eds Sreenivas, T. V. and Hari, K V. S), Signal processing and communications, SPCOM-93, Tata McGraw-Hill, New Delhi, 1993, pp. 129–134.
14. Moharir, P. S., *Geol Soc. India Mem*, 1992, **24**, 435–445.
15. Wallis, W D, Street, A P and Wallis, J S., *Combinatorics Room Squares, Sum-free Sets, Hadamard Matrices*, Springer-Verlag, Berlin, 1972
16. Brewer, J. W, *Trans IEEE*, 1978, **CAS-25**, 772–781
17. Moharir, P. S., *J Inst. Electron Telecomm Engrs.*, 1978, **24**, 351–354
18. Arrow, K J, *Social Choice and Individual Values*, John Wiley, New York, 1963.

ACKNOWLEDGEMENTS I am grateful to Dr H K Gupta, Director, NGRI for providing an environment conducive to such work and for permitting the publication of this paper Mr S. N. S Hussaini has converted my ambivalent scribble into a clean computer printout enhancing its readability, I am thankful to him. The figures were prepared by Mr M Kranthi Kumar very competently.

Received 1 March 1993, revised accepted 26 April 1994

INSA Medal for Young Scientists—1995

Instituted by the Indian National Science Academy in 1974 the Medal is awarded annually in recognition of outstanding work of scientists below the age of 32 (as reckoned on 31st December preceding the year of award). Only those born on or after 1 January 1963 are eligible for consideration in 1995. The work done in India by the nominee will be taken into consideration for the award.

The awardee is presented a **medal** and **a cash award of Rs 10,000**. In addition, the recipient is considered for a research grant by the Academy not exceeding Rs 40,000 per year.

Nominations for the awards for 1995 may be made by Fellows of the Academy, as also by the established scientific societies of all India character, University faculties and departments, or the research institutions. The last date for the receipt of nominations in the Academy is 15 November 1994.

Nomination proforma can be obtained from the Indian National Science Academy, Bahadur Shah Zafar Marg, New Delhi 110 002 by sending a self-addressed envelope of 25 cm × 12 cm size.