optical system. By selecting different pairs of mirrors it is possible to work with baselines in the range 5 to 640 metres.

The faintest star which SUSI can reach is expected to have a magnitude of + 8 or + 9, over 100 times fainter than the original instrument at Narrabri could reach, and the angular size range will be from 0.02 to 0.00005 seconds of arc. It has already worked with baselines of up to 80 m and in the near future I think we can look forward to great things.

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Bose statistics – Before and after*

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Quantum theory was born in October 1900 with Max Planck's discovery of the law of spectral distribution of the energy density of black-body radiation. Around June–July 1924, Satyendra Nath Bose from the Department of Physics at Dacca University sent to Albert Einstein a short four-page paper containing the first logically complete derivation of Planck's Law. Einstein had a great deal to do with recognizing the importance of Bose's work, having it published, and applying the idea elsewhere.

The story of what happened in the quarter century between Planck and Bose has been recounted on many occasions[1], and as the saying goes, to do so again might be 'as tedious as a twice-told tale, to vex the dull ear of a drowsy man'[2]. But the occasion of Bose's birth centenary is very special, making it well worth telling the tale again for a new generation of readers. In this spirit, I will try to describe the background to Bose's work, as a crowning achievement in a great chapter of physics, and convey its significance and impact. With no pretense to completeness, some of the personalities in the history of this subject will be recalled, and the principal events selectively and briefly recapitulated in chronological order.

Universal temperature radiation

1859–60. The story begins with Gustav Kirchoff of the University of Heidelberg, sometimes called the grandfather of the quantum theory. Kirchoff studied the properties of radiation in equilibrium with matter at a given common temperature, and on thermodynamic arguments proved the following basic result[3]:

$U(v, T)$ = energy density of radiation at temperature T per unit frequency interval at frequency v

= a universal function of v and T, independent of the nature of the matter emitting and absorbing radiation. (1)

Such radiation has come to be called 'black-body radiation', or sometimes also 'temperature radiation'. This *universality* concept due to Kirchoff is clearly a very fundamental one; it naturally directed both experimental and theoretical attention to the measurement and explanation of the function $U(v, T)$.

1879. Some two decades later, Josef Stefan experimentally measured the total energy density of radiation at temperature T, 'summing' over all frequencies, and found that it was proportional to the fourth power of the temperature[4]:

$$\int_0^\infty dv\, U(v, T) = \sigma T^4.$$ (2)

1884. A few years later, Ludwig Boltzmann was able to give a theoretical explanation of Stefan's findings[5]. He treated temperature radiation as a thermodynamic system on its own, and made use of the fact that the pressure exerted by radiation is one-third its energy density. This directly led to the result, equation (2), which is known as the Stefan–Boltzmann Law – and the constant σ is named after both of them too.

1893. Next we come to the ingenious theoretical analysis of Wilhelm Wien, who proposed what would today be called a thought experiment[6]. He considered temperature radiation contained in a spherical cavity with perfectly reflecting walls, and analysed the effect of slowly – adiabatically – reducing the radius of the cavity. In the process the temperature of the radiation

increases, while simultaneously radiation in each small frequency interval moves up to a slightly higher frequency interval as a result of reflection at the shrinking cavity walls. Applying thermodynamic principles to this situation, he proved a fundamental *scaling law*: for any real positive parameter a,

$$U(av, aT) = a^3 U(v, T). \tag{3}$$

This result is known as *Wien's Theorem*, and it simplified Kirchoff's universal function of two independent variables to another universal function of just the ratio of frequency to temperature:

$$U(v, T) = v^3 f(v / T). \tag{4}$$

Thus the challenge became that of finding the single function $f(\xi)$ of just one argument, $\xi = v/T$.

1896. Soon after, Wien followed up his exact result [equation (3)] with a more or less heuristic proposal[7] for the form of the function $f(\xi)$. For this he turned to the process of interaction between radiation and matter, and as a model for the latter he took a gas with the well-known Maxwellian velocity distribution at a temperature T:

$P(v, T)\, dv$ = probability of speed of gas molecules being in the interval v to $v + dv$

$$= (2/\pi)^{1/2} \cdot (m/kT)^{3/2} \cdot v^2 \exp(-mv^2/2kT). \tag{5}$$

(Here k is the Boltzmann constant.) Wien's idea was that there might be some relation between gas molecules of speed v and radiation of frequency v, v and v being related in some way, such that the former absorbed and emitted the latter. This was a qualitative idea, not the result of a detailed analysis of emission and absorption processes, but it suggested that the temperature dependences of the functions $P(v, T)$ and $U(v, T)$ should be similar.

Combined with the scaling theorem, equations (3, 4), this led to the proposal

$U(v, T)$ = (some function of v) exp [− (another function of v)/T]

$$= av^3 \exp(-bv/T),$$

$$f(\xi) = a \exp(-b\xi), \tag{6}$$

where a and b are some constants. This form for $U(v, T)$ is known as *Wien's Law*. Experimentally it was found to be well obeyed for large values of the ratio v/T, that is, for large ξ. For a given temperature this means that Wien's Law is true for high enough frequencies.

1899. We now come to the work of Max Planck, successor to Kirchoff at Berlin and the father of the quantum theory. He was naturally attracted to the problem first enunciated by Kirchoff, and for his attack he considered a model for matter quite different from the Maxwellian gas used by Wien. Planck represented matter by charged simple harmonic oscillators of various frequencies v – the so-called Hertzian oscillators. Each such oscillator is able to absorb and emit radiation only at its own natural frequency.

On the basis of the classical Maxwell equations of electromagnetism one can calculate the rates for these processes and one finds, as Planck did, the following results:

Radiation at frequency v *emitted* by Hertzian oscillator in unit time

$$= (8\pi^2 e^2 v^2)/(3mc^3)\varepsilon(v, T),$$

$\varepsilon(v, T)$ = average energy of oscillator at temperature T; (a)

radiation at frequency v *absorbed* by Hertzian oscillator in unit time

$$= (\pi e^2/3m) \cdot U(v, T) \qquad \text{(b)} \quad (7)$$

Here e and m are the charge and mass respectively of the oscillator, and v its frequency; and in deriving the absorption rate it is of course assumed that the oscillator is surrounded by temperature radiation. At equilibrium these two rates must be equal. This immediately leads to the Planck connection[8]

$$U(v, T) = (8\pi v^2 / c^3) \cdot \varepsilon(v, T) \tag{8}$$

which he announced on 18 May 1899. Notice that by this result Planck had transferred the problem of finding Kirchoff's function $U(v, T)$ referring to radiation to the problem of finding the average energy of a material oscillator of natural frequency v, given that it is at temperature T. Notice also that Planck did not at this stage invoke the equipartition theorem of classical statistical mechanics – had he done so, he could have set $\varepsilon(v, T) = kT$, and that would have immediately led him to the next result to which we now turn.

1900. Meanwhile Lord Rayleigh approached the problem along the lines of classical statistical mechanics, and the equipartition theorem. By counting the number of 'modes' – independent solutions – of the free electrodynamic field equations of Maxwell in each small frequency interval, and ascribing an energy kT to each mode, he found the result that read (after an amendment by James Jeans in 1905)[9]

$$U(v, T) = (8\pi v^2/c^3) kT,$$

$$f(\xi) = (8\pi k/c^3)(1/\xi). \tag{9}$$

This is the Rayleigh–Jeans (R-J) formula. Experimentally it is well obeyed for small ξ, i.e. for low

enough frequencies at any given temperature. In spirit, Rayleigh's derivation is similar to Boltzmann's work of 1884 – temperature radiation was treated as a thermodynamic system on its own, with no reference to interaction with matter. It was also soon realized that the R–J law is the *inescapable conclusion* of classical electrodynamics coupled with statistical mechanics. But it leads to a catastrophe: the total radiation energy density at a temperature T turns out to be infinite,

$$\int_0^\infty (8\pi v^2/c^3)\, kT\, dv = \infty \tag{10}$$

implying an infinite value for the Stefan-Boltzmann constant σ!

At this point we may list the Wien and the R–J expressions for the universal function $f(\xi)$ and Planck's energy expression $\varepsilon(v, T)$ experimentally valid in their respective limits:

	Wien limit large ξ, high v	R–J limit small ξ, low v
$f(\xi)$	$ae^{-b\xi}$	$\dfrac{8\pi k}{c^3} \cdot \dfrac{1}{\xi}$
$\varepsilon(v, T)$	$\dfrac{c^3}{8\pi} \cdot v \cdot a \exp(-bv/T)$	kT

$$\tag{11}$$

Of course there is the general connection following from Wien's Theorem and Planck's calculation:

$$f(v/T) = \frac{8\pi}{c^3} \cdot \frac{\varepsilon(v, T)}{v}. \tag{12}$$

Planck's law, photons and the Bohr atom

Sunday, 7 October, 1900. It appears that Planck had believed that the Wien Law (6) would be found to be experimentally valid for all v, and had hoped to derive it from basic thermodynamic principles. On the day and date mentioned above – incidentally, Niels Bohr's fifteenth birthday – the experimental physicist Heinrich Rubens and his wife visited the Plancks for tea in the afternoon. During conversation, Rubens told Planck of his and Ferdinand Kurlbaum's latest experimental measurements of the spectrum of temperature radiation at the low frequency end, and that the Wien Law fails there; the results agreed on the other hand with a linear temperature dependence!

Planck saw immediately that he had to find a bridge – an interpolation – between these two limits, an expression for $\varepsilon(v, T)$ to smoothly connect the two ends of high v and low v, and set to work to do so. However instead of working directly with the energy $\varepsilon(v, T)$ he

switched to the entropy S of the material oscillator. The argument went along these lines. Starting with expressions for S and ε in terms of T, and eliminating T, one ends up with S expressed as a function of ε, and one then has the thermodynamic relation

$$dS/d\varepsilon = 1/T = \text{function of } \varepsilon. \tag{13}$$

Now the two limiting expressions become:

Wien limit: high v or small ε:

$$\frac{dS}{d\varepsilon} = \frac{-\ln \varepsilon}{bv} + \frac{1}{bv} \ln\left(\frac{avc^3}{8\pi}\right),$$

$$\frac{d^2S}{d\varepsilon^2} = \frac{-1}{bv\varepsilon}; \tag{a}$$

R–J limit: low v or large ε:

$$dS/d\varepsilon = k/\varepsilon,$$

$$d^2S/d\varepsilon^2 = -k/\varepsilon^2 \tag{b} \tag{14}$$

Planck interpolated these two limiting expressions for $d^2S/d\varepsilon^2$ by *adding the denominators in the denominator*:

$$\frac{d^2S}{d\varepsilon^2} = -1 \Big/ \left(bv\varepsilon + \frac{\varepsilon^2}{k}\right)$$

$$= -k/\varepsilon(\varepsilon + kbv). \tag{15}$$

A single integration leads to

$$\frac{dS}{d\varepsilon} = \text{constant} + \frac{1}{bv} \ln\left(1 + \frac{kbv}{\varepsilon}\right). \tag{16}$$

Imposition of the R–J limit when ε is large and equal to kT shows this constant to be zero, so

$$\frac{dS}{d\varepsilon} = \frac{1}{T} = \frac{1}{bv} \ln\left(1 + \frac{kbv}{\varepsilon}\right). \tag{17}$$

Setting $kb = h$, a new constant, we get the results:

$$\varepsilon(v, T) = hv/(\exp(hv/kT) - 1),$$

$$U(v, T) = \frac{8\pi v^2}{c^3} \cdot \frac{hv}{\exp(hv/kT) - 1},$$

$$f(\xi) = \frac{8\pi h}{c^3} \cdot \frac{1}{\exp(h\xi/k) - 1},$$

$$a = 8\pi h/c^3, \quad b = h/k. \tag{18}$$

This is the celebrated Planck radiation law, and thus in the space of a few hours was quantum theory born! But at this stage there was no hint yet of quanta or quantization of energy.

Planck publicly announced this result on 19 October 1900, and published it soon after[10].

14 December 1900. In the weeks and months that followed, Planck tried to give a theoretical justification for his expression (18) for the average energy of a material oscillator at temperature T, an expression obtained by a simple yet inspired mathematical interpolation. In vain did he search among Boltzmann's papers for a statistical argument that might lead to the desired result. Finally, 'as an act of desperation.... to obtain a positive result, under any circumstances and at whatever cost', he resorted to the concept of irreducible packets or quanta of energy – a stroke and a measure of his genius – and in mid-December, 1900, he presented the following statistical derivation[11].

Assume a large number, N, of identical (but distinguishable) material oscillators, which all together share a certain total energy E; and further assume that this energy comes in P individual packets of amount ε_o each. Let ε be the average energy per oscillator, we have the relations

$$E = P\varepsilon_o = N\varepsilon,$$

$$P = N\varepsilon/\varepsilon_o \tag{19}$$

Now, *on the assumption that these packets of energy are in no way distinguishable from one another*, the number of ways the P packets can be distributed over the N oscillators is

$$W = \text{coefficient of } x^P \text{ in } (1 + x + x^2 + \cdots)$$
$$\times (1 + x + x^2 + \cdots)\cdots(1 + x + x^2 \cdots), N \text{ factors,}$$
$$= \text{coefficient of } x^P \text{ in } (1 - x)^{-N}$$
$$= (N + P - 1)!/P!(N - 1)! \tag{20}$$

Here in the first line we have one factor for each of the N (distinguishable!) oscillators. Using the Stirling formula and the Boltzmann–Planck connection between entropy S and the statistical probability W, one has for the average entropy per oscillator

$$S/N = (k/N) \ln W.$$

$$\simeq k\left[\left(1 + \frac{\varepsilon}{\varepsilon_o}\right)\ln\left(1 + \frac{\varepsilon}{\varepsilon_o}\right) - \frac{\varepsilon}{\varepsilon_o}\ln\frac{\varepsilon}{\varepsilon_o}\right]. \tag{21}$$

The connection to the temperature is provided by eq. (13) (with S/N in place of S there), so

$$\frac{d}{d\varepsilon}\frac{S}{N} = \frac{k}{\varepsilon_o}\ln\left(1 + \frac{\varepsilon_o}{\varepsilon}\right) = \frac{1}{T},$$

i e. $\varepsilon(v, T) = \varepsilon_o/(\exp(\varepsilon_o/kT) - 1)$. $\tag{22}$

Now, agreement with Wien's theorem (12) demands ε_o be proportional to the oscillator frequency v; so setting $\varepsilon_o = hv$ for a constant h, we end up with

$$\varepsilon(v, T) = hv/(\exp(hv/kT) - 1). \tag{23}$$

Planck's genius showed both in the interpolation (15) leading to the result (18), and in his 'packets of energy' concept that then supplied a theoretical derivation As we shall see, his method of counting W based on indistinguishable packets of energy is the same as Bose's idea of 1924, but this was not clearly realized at the time. Of course Planck applied it in the context of material oscillators. The first explicit statement that in Planck's method the oscillators are distinguishable while the energy packets are not, is due to Ladislas Natanson in 1911.

1905. A major step forward was taken with Einstein's analysis of Planck's Law, and the birth of the photon concept[12]. In an effort to bring out the departures from classical ideas, Einstein studied the nonclassical high frequency Wien limit of Planck's Law, treating that portion of radiation as a thermodynamic system on its own. If we have a range of frequencies Δv (in the Wien region) and a spatial volume V, the total energy of the radiation is

$$E \simeq V \cdot \frac{8\pi v^2}{c^3} \cdot hv \cdot \exp(-hv/kT) \cdot \Delta v$$
$$= \mathcal{N} \cdot V \cdot \exp(-hv/kT),$$
$$\mathcal{N} = 8\pi hv^3 \Delta v/c^3. \tag{24}$$

We may choose either E and T or E and V as independent thermodynamic variables. Choosing the latter, the entropy of the above portion of radiation turns out to be (upon integration of (13)):

$$S(E, V) = \frac{kE}{hv}\left(1 - \ln\frac{E}{\mathcal{N}} + \ln V\right). \tag{25}$$

Keeping the total energy E fixed, if we compare two spatial volumes V_1 and $V_2 < V_1$, the difference in entropy is

$$S(E, V_2) - S(E, V_1) = \frac{kE}{hv}\ln(V_2/V_1). \tag{26}$$

Now Einstein turned to the Boltzmann Planck relation (21) – $S = k \ln W$ – and 'read it backwards', something no one had done till then! Thus he was able to relate the two statistical probabilities:

$$k \ln\left(\frac{W_2}{W_1}\right) = \frac{kE}{hv}\ln\left(\frac{V_2}{V_1}\right),$$

i.e. $\quad\dfrac{W_2}{W_1} = \left(\dfrac{V_2}{V_1}\right)^{E/hv}. \tag{27}$

The interpretation now is this: if an amount of energy E of radiation at frequency v is known to be contained in a spatial volume V_1, and one asks for the probability of its being found in a subvolume V_2 less than V_1, the answer

is given by the right hand side of (27). But this is precisely the kind of result we expect if we had a uniformly distributed gas of 'classical independent' molecules in a volume V_1: if the number of gas molecules is n, then the probability of all of them being simultaneously found in a subvolume V_2 is $(V_2/V_1)^n$. So Einstein drew the conclusion that in the nonclassical Wien limit, radiant energy at frequency v appears like a gas of individual localized packets, each of energy hv. This was how the photon concept was arrived at, and the photoelectric effect was put forward as a way to test the idea.

1909 The next major step was due again to Einstein, but *this time he exploited the complete Planck Law to calculate the fluctuations in the energy of radiation*[13]. Assuming a unit spatial volume and a small frequency range Δv, the energy of temperature radiation is

$$E = \frac{8\pi v^2 \Delta v}{c^3} \cdot hv \,/\, (\exp(hv/kT) - 1). \qquad (28)$$

Now it is a general result of statistical thermodynamics that the fluctuation in energy, ΔE, is given by

$$(\Delta E)^2 = kT^2 (\partial E/\partial T), \qquad (29)$$

so using (28) one finds

$$(\Delta E)^2 = \frac{c^3}{8\pi v^2 \Delta v} E^2 + hvE. \qquad (30)$$

Here is the interpretation; the first term is the result of a pure R–J distribution, so it is the *classical wave contribution*; the second term is the result of a pure Wien distribution, so it is the *nonclassical photon* or *particle contribution*. Indeed, in the latter limit we saw that $E = nhv$, where n is the number of photons, so the above fluctuation result would read $(\Delta n)^2 = n$: this is just the consequence of Poisson statistics appropriate for classical distinguishable particles.

The lesson drawn was that radiation is partly like classical waves (specially at low frequencies), partly like distinguishable particles (especially at high frequencies), but in reality it is not entirely either! One cannot but be struck by the deep and extraordinarily insightful ways in which Einstein repeatedly 'played' *with Planck's Law to extract its many consequences. It* is also instructive to remark that the formula (30) is basically the same as the Planck interpolation (15): one is just the inverse of the other!

One last stop in our journey before Bose enters the story.

1917 In 1913, Niels Bohr had shown the connection between Planck's constant and the mechanics of the atom. This had led to a new description for matter, radically different from classical conceptions, involving discrete stationary states and the emission or absorption

of radiation to accompany transitions between them. Now Einstein returned to Planck's Law, and attempted to derive it on the basis of radiation interacting with matter treated according to Bohr's theory; in the process he introduced his famous A and B coefficients[14]. For simplicity consider matter with just two quantized Bohr levels, m and n say, with energies ε_m and ε_n, the former being higher. Bohr's frequency condition determines the frequency of emitted or absorbed radiation:

$$\varepsilon_m - \varepsilon_n = hv. \qquad (31)$$

Applying Boltzmann statistics to the atoms of matter the relative populations of the two levels are in the ratio of $\exp(-\varepsilon_m/kT)$ to $\exp(-\varepsilon_n/kT)$. Now Einstein considered three kinds of transitions: if $\rho(v)$ is the energy density of radiation at frequency v, there is an upward transition rate $n \to m$ due to absorption of radiation, given by

$$\rho(v) B_n^m \exp(-\varepsilon_n/kT) \qquad (32)$$

and involving a B-coefficient; there is a spontaneous downward transition rate $m \to n$ accompanied by emission of radiation, given by

$$A_m^n \exp(-\varepsilon_m/kT) \qquad (33)$$

and involving an A-coefficient; and finally there is an induced or stimulated downward transition rate $m \to n$, given by

$$\rho(v) B_m^n \exp(-\varepsilon_m/kT) \qquad (34)$$

involving another B-coefficient. In equilibrium, the expression (32) must equal the sum of expressions (33) and (34). This gives using (31):

$$\rho(v) = A_m^n \,/\, (B_n^m \exp(hv/kT) - B_m^n). \qquad (35)$$

Now Einstein essentially argued as follows: (i) if there were no stimulated emission, $B_m^n = 0$, we would end up with Wien's Law (6) rather than Planck's, so $B_m^n \neq 0$; (ii) for high temperatures, we must recover the R–J Law (9) with $\rho(v)$ proportional to T, so we must have the equality $B_n^m = B_m^n$; (iii) agreement with Wien's Theorem (4) then fixes the ratio of the A and the B coefficients:

$$A_m^n \,/\, B_m^n = 8\pi \, hv^3/c^3. \qquad (36)$$

Putting this into (35) gives us Planck's Law!

We can appreciate that this magnificent analysis of Einstein is actually a drawing together of many distinct ideas: the Bohr model and Boltzmann statistics for matter, the A and B coefficients for downward and upward transitions, and the constraints of agreeing with the R–J Law at high temperatures and with Wien's Theorem. In the end, we learn a great deal about matter and the need for and relations between the A's and B's; in all this there was no reference to classical Maxwellian electromagnetism.

Recapitulating the above developments one sees many swings in the methods of analysis between treating temperature radiation as a thermodynamical system as its own, and studying its interaction and equilibrium with matter. Kirchoff's original insight had pointed to the universality of the properties of radiation, and Boltzmann's work had treated radiation as a system by itself. Wien's scaling theorem and law had invoked the process of interaction with matter, and used a particular model for the latter. The R–J Law was based on a study of radiation by itself. With Planck there was a definite switch to interaction with matter, indeed he transferred the problem from radiation to matter! In 1905 and 1909 Einstein extracted consequences of Planck's Law, dealing with radiation on its own; by 1917 he turned to the processes of interaction with matter as described by the Bohr model.

New Year's day 1894

Now we turn to a brief life sketch of Satyendra Nath Bose, born in Calcutta on New Year's day 1894 to Surendra Nath Bose, a railway engineer, and Amodini Devi. He was an only son, and had six younger sisters. He studied in the Hindu High School until matriculation in 1909, then went on to the Presidency College for the Intermediate Science course, then the B.Sc. degree, followed by the M.Sc. in 'mixed mathematics' in 1915. Among Bose's teachers at College were J. C. Bose and P. C. Ray; and among his class-mates and near contemporaries were M. N. Saha, P. C. Mahalanobis and J. C. Ghosh. As Churchill might have said – 'some teachers, some students'. Bose obtained the first rank in B.Sc. as well as in M.Sc., while Saha stood second both times.

Around 1916 the Vice Chancellor of Calcutta University, Sir Asutosh Mukherjee, had set up the University College of Science and introduced a programme of post-graduate teaching in physics and applied mathematics. Bose had joined the University as a research student in 1916. The next year, 1917, Sir Asutosh offered both him and Saha posts of Lecturers in the newly established Departments; and the two took it upon themselves to teach all the principal areas of physics and applied mathematics to the post-graduate students.

After four years at Calcutta, Bose moved in 1921 to the newly founded University of Dacca as a Reader in the Physics Department. This became his home for the longest part of his professional life, becoming Professor and Departmental Head in 1927. In 1945 Bose came back to Calcutta as Khaira Professor of Physics at the University, which position he held until retirement in 1956. Then for a brief two-year period he was Vice-Chancellor at Visvabharati, during which time he unsuccessfully tried to introduce teaching and research in the sciences. At Pandit Nehru's urging (we are told),

Bose served as a member of the Rajya Sabha from 1952 to 1958. Thanks to being nominated by Paul Dirac, he was elected a Fellow of the Royal Society of London in 1958. He passed away on 4 February 1974, shortly after his eightieth birthday.

Bose was fluent in many languages – Bengali, English, French, German and Italian. He was a connoisseur of classical music, and played the 'Esraj' like a master. He also had a deep interest in the fine arts. In his later years he worked ceaselessly to popularize science by propagating it in Bengali. Truly a universalist and a deeply cultured personality.

Bose's earliest scientific paper was coauthored with Saha and appeared in the *Philosophical Magazine* in 1918. In 1920 the two class fellows translated the original papers of Einstein and Minkowski on special and general relativity from German into English, and this was published as a book by Calcutta University under the title 'The principle of relativity'. It is noteworthy that this is the earliest ever such translation worldwide. Soon after his great work of 1924, Bose spent the two-year period October 1924 to late 1926 in Europe, on leave of absence from Dacca University. He stayed in Paris for about a year, and then moved on to Berlin, before returning to Dacca. This was the period of the birth of quantum mechanics, a major revolution in physics. Due to a combination of factors, however, Bose was not able to participate in these developments at all. There are no scientific papers resulting from this stay in Europe, as Bose spent much of his time learning new experimental techniques in order to set up good quality laboratories in his own Department. In any event, Bose was not a prolific author of papers, his total output being about thirty!

Now we turn to Bose's epochal work.

Bose statistics

1924. Around March 1924, Saha visited Bose at Dacca. During conversations Bose told him of his sense of dissatisfaction with all the existing attempts at deriving Planck's Law, something he particularly felt while teaching the subject to his students. In response, Saha mentioned a 1923 paper of Einstein and Ehrenfest[15], which improved upon Einstein's 1917 work on the A and B coefficients by extension to multilevel atoms; and a 1923 paper of Pauli[16] which took into account the Compton effect involving radiation and electrons, and then examined the problem of stability of Planck's Law.

Against the background of his expressed sense of dissatisfaction, and information on more recent attempts, by June 1924 Bose produced his own entirely new and novel derivation of Planck's Law. Its main features were that it was internally consistent, logically satisfactory, made no use of the classical Maxwell theory of electromagnetism, made no reference to the

processes of interaction of radiation with matter, and treated temperature radiation as a system on its own, *as a gas of photons* Referring to the many existing treatments, Bose says in his paper:

'In all cases the derivations do not appear to me sufficiently justified from a logical point of view. Contrary to that it appears to me that the light quantum hypothesis in connection with statistical mechanics... should be sufficient for the derivation of the law independently of the classical theory'.

The above reconstruction of events is one possible version of what transpired and led to Bose's work. This is the version, incidentally, mentioned by Saha himself in an article he wrote soon after Einstein's passing away. There is, however, another reading of events very persuasively argued by John Stachel[17], who has had direct access to the Einstein archives. In Stachel's view, Bose must have completed his derivation of Planck's Law by the end of 1923; and the conversations with Saha around March 1924 probably started off fresh ideas relating to the interaction between matter and radiation. Bose first sent his brief four-page paper — 'Planck's Formula and Light Quantum Hypothesis' – to the *Philosophical Magazine*, which journal apparently did not respond for a long time. He then sent it to Einstein on 4 June 1924, saying in his covering letter: 'I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it... If you think the paper worth publication I shall be grateful if you arrange for its publication in *Zeitschrift fur Physik....*'. As Stachel suggests, it is highly unlikely that Bose would have risked allowing his paper to be under consideration in two journals more or less simultaneously, as acceptance by both would have caused a scandal that would have hurt him; therefore the submission to the *Philosophical Magazine* must have occurred well before Bose decided to approach Einstein.

As is well known, Einstein immediately recognized the significance of Bose's ideas. He translated the paper into German, and added at the end: 'Comment of the translator: Bose's derivation of Planck's formula appears to me to be an important step forward. The method used here gives also the quantum theory of an ideal gas, as I shall show elsewhere'. Einstein's translation of Bose's paper was received by the *Zeitschrift fur Physik* on 2 July 1924, and it was published in the August 1924 issue[18]. Considering that Bose mailed his article to Einstein on 4 June, and making allowance for non airmail transit times, this was fast work indeed.

Here now is a brief outline of Bose's derivation. Consider radiation at temperature T enclosed in unit spatial volume, $V = 1$. Let the total energy be E, and regard the radiation as a gas of photons, of various frequencies v, energies hv, and momenta hv/c.

(Remember here that Einstein's photon concept proposed in 1905 was initially not at all widely accepted, and had to wait till the discovery of the Compton effect in 1923 before gaining general support; also that the assignment of a definite momentum to a photon was made by Einstein only around 1916.) Divide the set of all possible frequencies into small intervals of widths Δv_s: the sth interval goes from, say, v_s to $v_{s+1} = v_s + \Delta v_s$. Photons in the sth interval have energies $E = hv_s$ and (magnitude of) momenta $p = hv_s/c$. The total phase space volume occupied by these photons is then

$$V\,(=1) \times 4\pi p^2 dp \text{ (shell in momentum space)}$$
$$\times 2(\text{independent states of polarization})$$
$$= (8\pi v_s^2 / c^3) \cdot \Delta v_s \cdot h^3. \qquad (37)$$

So, by the Planck rule of 1911 that the elementary cell in phase space for one degree of freedom is of extent h, the number of distinct and distinguishable single photon states contained in this phase space volume is

$$A_s = 1/h^3 \times (\text{above phase space volume})$$
$$= 8\pi v_s^2 \cdot \Delta v_s / c^3. \qquad (38)$$

This is independent of Planck's constant, and is precisely the 'first factor' in the Planck Law[18]! Bose regarded his derivation of this factor as particularly significant, since he remarked in his covering letter to Einstein: '...you will see that I have tried to deduce the coefficient $8\pi v^2/c^3$ in Planck's Law independent of classical electrodynamics, only assuming that the elementary regions in the phase-space has the content h^3....' We shall return later to comment on the factor of two in eq. (37) referring to photon polarization.

Next let us turn to Bose's statistical derivation of the 'second factor' in the Planck Law [eq. (18)]. Suppose there are N_s photons distributed over the A_s single photon states in the sth frequency range Δv_s. In how many ways can this distribution be done, and how must the counting be carried out? Bose introduced the following procedure: out of the A_s single photon states, let p_0^s be the number of states having no photon at all; p_1^s the number of states having one photon each; p_2^s the number of states having two photons each; and so on. Then the number of distributions of N_s photons over A_s states for given $p_0^s, p_1^s, \ldots$, is

$$W_s = A_s! / p_0^s! p_1^s! p_2^s! \ldots \qquad (39)$$

The key idea here is that it does not matter 'which' of the N_s photons goes into 'which' state, all that is relevant is the 'occupation number' in each of the A_s states. Stated in another way, we only ask in how many ways we can pick p_0^s states out of the A_s available ones, in which to put no photons at all; then p_1^s states out of

the remainder, in which to put one photon each; and so on. The constraints on the numbers $p_0^s, p_1^s, \ldots$ are evidently

$$\sum_{r=0, 1, \ldots} p_r^s = A_s,$$

$$\sum_s {}^s N_s h v_s \equiv \sum_s h v_s \sum_{r=0, 1, \ldots} r p_r^s = E. \quad (40)$$

No constraint on the total number of photons was invoked. The statistical probability of the entire 'complexion' is

$$W = \prod_s W_s. \quad (41)$$

Bose then proceeded to maximize W with respect to variations in the p_r^s which obey the two constraints (40); and this led him immediately and easily after identifying the temperature to the second Planck factor:

$$N_r = A_s / (\exp(h v_s / kT) - 1). \quad (42)$$

The combinatorics leading to Bose's expression (39) can be carried out in another entirely equivalent but superficially different looking way, as shown by Einstein[19], and this just takes us back to Planck's counting of 1900 given in eq. (20)! Namely, if N_s photons sans identity are to be distributed over the A_s single photon states, and these states are distinguishable one from another because they correspond to distinct directions of momentum and/or polarization, the number of ways this can be done is just

$$(N_s + A_s - 1)! / N_s! (A_s - 1)! \quad (43)$$

The P indistinguishable energy packets of Planck's calculation become here the N_s indistinguishable photons; and the N distinguishable material oscillators of Planck become the A_s distinct states. Thus while the two contexts are quite different, the calculations of statistical probabilities are exactly the same!

For completeness let us present now the variational calculation to maximize the statistical probability in the form (43). We have for a given complexion $\{N_s\}$:

$$W(\{N_s\}) = \prod_s (N_s + A_s - 1)! / N_s! (A_s - 1)!,$$

$$\sum_s N_s h v_s = E. \quad (44)$$

(With the slightly simpler counting compared to Bose's original calculation, we deal directly with the N_s rather than the p_r^s, so we have just one energy constraint in (44) replacing the two constraints in (40).) Maximizing W subject to the energy constraint (after use of Stirling's formula) leads to:

$$\delta \ln W = 0 \Rightarrow \sum_s \ln (1 + A_s / N_s) \cdot \delta N_s = 0,$$

$$\delta E = 0 \Rightarrow \sum_s h v_s \cdot \delta N_s = 0. \quad (45)$$

The use of a Lagrange multiplier λ yields

$$N_s = A_s / (e^{\lambda h v_s} - 1). \quad (46)$$

Switching now to the familiar notations $v_s \to v$, $v_{s+1} \to v + \Delta v$, we find:

$U(v, T) \Delta v$ = energy density of radiation in the frequency range Δv,

$$= N_s \cdot h v_s$$

$$= \frac{8 \pi v^2}{c^3} \cdot \Delta v \cdot \frac{h v}{e^{\lambda h v} - 1}. \quad (47)$$

By an entropy calculation one shows that $\lambda = 1/kT$, so one ends up with Planck's Law!

The calculation is ultimately 'elementary', the novelty lies in the *exclusive* use of the photon concept, and in the method of computing the statistical probability W. But here, it seems, is a classic case of Arthur Koestler's 'sleepwalker' phenomenon! As Bose recalled much later[20]: 'I had no idea that what I had done was really novel. I thought that perhaps it was the way of looking at the thing. I was not a statistician to the extent of really knowing that I was doing something which was really different from what Boltzmann would have done from Boltzmann statistics. Instead of thinking of the light quantum just as a particle, I talked about these states. Somehow, this was the same question that Einstein asked when I met him. How had I arrived at this method of deriving Planck's formula? Well, I recognized the contradictions in the attempts of Planck and Einstein, and applied the statistics in my own way, but I did not think that it was different from Boltzmann statistics'.

Well – but it *was* different and new – not Boltzmann statistics, but the birth of quantum statistics, before the birth of quantum mechanics itself. Going back to the Einstein energy fluctuation formula (30) of 1909 *one can say: radiation is partly like classical waves, and partly like classical independent distinguishable particles; but truly and in essence it is a system of indistinguishable Bose particles, or quantized waves* – these are one and the same. Remembering that on the one hand Einstein had come upon the photon concept by analysing only the Wien limit of Planck's Law, in which limit indistinguishability plays no role and the photons are statistically independent[21]; that on the other hand he had based the fluctuation formula on the complete Planck Law; and that for twenty years he had tried unsuccessfully to bridge this conceptual gap and to derive Planck's Law from his photon concept; one can understand how he was mentally primed to immediately

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grasp the significance of Bose's work, and to understand the kind of 'statistical nonindependence' of photons that the complete Planck Law implied and that had eluded him for so long.

Now the promised comment on the factor of two in Bose's calculation (37) of the photon phase space volume. In the published version of the paper, after translation and (presumably) some editing by Einstein, the relevant sentence reads: 'In order to take into consideration the fact of the polarization it appears necessary to multiply this number by 2 so that we obtain...'. But it seems that in the original manuscript Bose sent to Einstein in June 1924, he had dwelt on this point at greater length, and had ascribed a 'helicity', or angular momentum in the direction of momentum, with possible values $\pm h/2\pi$ to each photon. Einstein evidently found this idea too revolutionary and omitted it during translation – such was the state of understanding of these matters in those days. Fortunately for us, Raman and Bhagavantam refer explicitly to this in their 1931 paper in these words[22]: 'In his well-known derivation of the Planck radiation formula from quantum statistics, Prof. S. N. Bose obtained an expression for the number of cells in phase space occupied by the radiation, and found himself obliged to multiply it by a numerical factor 2 in order to derive from it the correct number of possible arrangements of the quantum in unit volume. The paper as published did not contain a detailed discussion of the necessity for the introduction of this factor, but we understand from a personal communication by Prof. Bose that he envisaged the possibility of the quantum possessing besides energy $h\nu$ and linear momentum $h\nu/c$ also an intrinsic spin or angular momentum $\pm h/2\pi$ round an axis parallel to the direction of its motion'.

The ideal quantum gas, quantum mechanics and field theory

1924–26. Events unfolded with dramatic speed in the months and years following publication of Bose's paper. As promised in his translator's remark, Einstein applied Bose's procedure to work out the theory of the ideal quantum gas[23]. Here, as in the nonrelativistic regime the total number of massive particles is conserved, the chemical potential comes in. Einstein again derived a fluctuation formula, now for the particle density ρ. Comparing the earlier and the later results:

$$\text{Radiation:} (\Delta E)^2 = \cdots E^2 + \cdots \quad E, \quad \text{(a)}$$

$$\underset{\substack{\downarrow \\ \text{classical R–J} \\ \text{wave term}}}{} \quad \underset{\substack{\downarrow \\ \text{quantum Wien} \\ \text{particle term}}}{}$$

$$\text{Matter:} (\Delta\rho)^2 = \cdots \rho^2 + \cdots \rho, \quad \text{(b)}$$

$$\underset{\substack{\downarrow \\ \text{quantum} \\ \text{wave term}}}{} \quad \underset{\substack{\downarrow \\ \text{classical} \\ \text{particle term}}}{}$$

$$\text{(48)}$$

we see a switch of the old and the new. Soon after getting this result, Einstein asked for and went through de Broglie's thesis: here was an independent indication of the wave aspect of matter, and it became an important motivating factor for Schrödinger in developing wave mechanics, with origins in Bose's work!

In 1925 Pauli announced the exclusion principle[24], and in early 1926 Fermi made it the basis for another kind of quantum statistics, different from Bose's[25]. Later in 1926, Dirac succeeded in deriving both forms of quantum statistics from the two possible kinds of permutation symmetry for the wave functions of assemblies of identical particles in quantum mechanics[26]. Actually, when Heisenberg's matrix mechanics was discovered in summer 1925 while Bose was still in Europe, Einstein posed two problems to Bose: first, to analyse the excess correlations among photons implied by indistinguishability[27]; second, to link his statistics with the new quantum mechanics, and compute the transition probabilities of radiation. But Bose was to do neither.

1927. This was the year when Dirac applied the principles of canonical quantization to the classical Maxwell field, thereby inaugurating quantum field theory[28]. Thus he brought to a close the cycle of developments starting with Planck in 1900 and culminating with Bose in 1924. After Dirac's work, Bose statistics was seen to be an automatic consequence of field quantization followed by a particle interpretation, and Planck's Law became an automatic result of the canonical distribution for the quantized field. In the Dirac commutation relation $-[a, a^+] = 1 -$ for photon annihilation and creation operators, it is the unity on the right hand side that accounts for Einstein's A coefficient for spontaneous emission.

, At this stage, after field quantization, one saw that the abstract field operator and its state are the primary quantities; while the particle language is but one way of talking about the configurations of the field. Interchange of identical particles implies no change in the condition of the field, so it leads to no observable effects at all.

1940. Many years were to pass before Pauli found a connection between the spin of an elementary object, and the kind of statistics, Bose–Einstein or Fermi–Dirac, that it had to obey[29]. In his classic paper he was able to show that as a consequence of the requirements of locality and positive-definite energy in the framework of relativistic quantum field theory, more precisely for free particles obeying linear relativistic wave equations, integer spin particles had to obey Bose statistics while

half-odd integer spin particles were governed by the exclusion principle.

Towards modern times

Every known particle today is either a boson or a fermion. These names were coined by Dirac. Combining Pauli's theorem with angular momentum theory in quantum mechanics, it is easy to see that bosons can be produced as bound states or composites of an even number of fermions. Thus for example both deuterons and alpha particles behave as bosons, as long as one restricts oneself to processes in which the energy is low enough so that the composite systems or bound states do not break up.

One would be tempted to think that, in contrast, fermions are irreducible, and cannot be made except out of other fermions (plus of course bosons). Strangely, however, there are situations where fermions can in principle arise purely out of bosons! One is the system of a magnetic monopole and an electric charge, both spinless, obeying the Dirac quantization condition in its simplest form[30]. Then the total angular momentum of the combined system becomes half-odd integral, and this is caused by the interaction. There is, however, no bound state possible in this case. Another is a nonlinearly coupled system of boson fields – the Skyrme model – where, as a result of the nontrivial topology of certain solutions, the total system behaves as a fermion[31].

There are numerous examples of bosons and manifestations of Bose statistics in elementary particle physics. One thinks of the hordes of mesons – $\pi, \eta, \rho, K, K^*, K^0, \overline{K}^0, \omega, \phi, B, D, \cdots$ and more recently the illustrious cousins of the photon, the $W^{\pm}$ and Z bosons, which are carriers of the electroweak forces[32]. More generally, it appears that while the basic categories of matter – quarks and leptons – are fermionic in nature, the carriers of the fundamental interactions – photons, W and Z, gluons, gravitons – are all bosonic. Bose statistics leads to important selection rules governing possible states and decay processes of particles: a spin one particle cannot decay into two identical spinless particles; two pions of the same charge must have even relative angular momentum; . . . Beyond these, in reactions described by the S-matrix, there are important relations referred to as 'crossing symmetry relations': processes related by interchange of incoming and outgoing particles are connected by suitable analytic continuation in energy, scattering angle, etc. This may be regarded as a generalized manifestation of Bose symmetry. The underlying reason is that quantized fields as a whole, containing both possibilities of destruction of particles and creation of antiparticles (or vice versa) are involved in dynamics of particle processes; these parts can never be split off or separated from one another. The most familiar case of crossing symmetry,

however, relates to fermions: thus Bhabha (e^+e^-) and Moller (e^-e^-) scattering get connected or determine one another by analytic continuation.

In the physics of many-body systems, Bose condensation is a much discussed consequence of Bose–Einstein statistics. The classical notion of independent noninteracting particles implies a tendency to stay uncorrelated, to 'go it alone'. Bosons on the other hand, even in the absence of any specific forces, display a tendency to stick to one another due merely to their identity, to 'do as the others do'. Thus both the functioning of the laser, dependent on Einstein's stimulated emission process, and the Hanbury-Brown-Twiss effect are traceable to this delicate quality of bosons. For theoretical analysis of complex systems too, bosons come in handy. Thus they are often introduced as a convenient or economical shorthand called 'effective bosons' to deal with pairs of fermions, and the technique itself is sometimes referred to as 'bosonization'. Cooper pairs in the conventional theory of superconductivity are an example. Similarly in low-to-medium energy nuclear physics – where the principal actors are the proton and neutron, both fermions – we have the unexpectedly successful 'Interacting Boson Model'. And finally we have the Higgs and Goldstone bosons – cause and consequence respectively of spontaneous breaking of continuous symmetry.

Bose's name has clearly become part and parcel of contemporary physics.

Comparisons and conclusions

We saw that the strongest motivating factor behind Bose's work was a feeling of dissatisfaction with existing derivations of Planck's Law, and a desire for a clean approach which would be logically acceptable. The photon idea had come from the Wien limit of Planck's Law; Bose showed how with his new statistics the photon idea led transparently to the complete Planck Law. But it was unfortunate that he could not participate in later dramatic applications of his discovery.

It is interesting to compare this situation with the motivations behind another great, slightly later, Asian theoretical discovery – Yukawa's theory of mesons as carriers of nuclear forces, dating from 1934. As Nambu describes the situation[32]: 'The greatness of Yukawa can be characterized, I think, by his ability to force the (above) issues. Simply put, he believed in the correctness and the reality of the two beautiful theories of the twentieth century, relativity and quantum mechanics, which had succeeded so brilliantly, and he applied the consequences of these theories directly to the atomic nucleus'. In both cases we see a sense of directness and great self-confidence. Indeed Yukawa himself has said[13]: 'Looking back, I was strangely full of self-confidence in the autumn of 1934 when I arrived at the

existence of mesons as the consequence of the theory of nuclear forces'.

The pioneering Indian contributions in physics came in 1920, 1924 and 1928. Those from Japan were slightly later – Yukawa in 1934, Tomonaga in the forties. Both were totally rooted in their own cultures and educational systems, though in the case of Japan the return of Nishina from Europe helped build up the remarkable Japanese school around Sakata, Umezawa, Nishijima and Nakano to name a few. The strength of cultural identity is expressed beautifully when Nambu says of Tomonaga[32]: 'He possessed fine Japanese sensibilities, and the year he spent in USA (1949 in Princeton) seems to have been a hardship. 'I have been exiled to heaven,' he complained to his students. Bose's character and sensibilities, one gathers, were similar.

To end on a lighter vein, one must point out how easily even professional physicists and historians of physics forget who exactly Bose was. In his book Nambu refers repeatedly to M. K. Bose[32]! And in a recent book on the history of radiation theory[34], S. N. Bose is throughout referred to as J. C. Bose (with the latter's years of birth and passing). '*Sic transit gloria mundi.*'

Abraham Pais has given Bose's 1924 paper the honour of being the fourth and last revolutionary paper of the Old Quantum Theory – the first three are Planck's 1900 paper, Einstein's 1905 photon paper, and Bohr's 1913 paper on atomic structure[35]. This is exalted company indeed.

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