On singularity-free cosmological models

N. Dadhich, R. Tikekar* and L. K. Patel**

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411 007, India
*Permanent address: Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 358 120, India
**Permanent address: Department of Mathematics, Gujarat University, Ahmedabad 380 009, India

In 1990 Senovilla[1] obtained an interesting cosmological solution of Einstein's equations that was free of the big-bang singularity. It represented an inhomogeneous and anisotropic cylindrical model filled with disordered radiation, $\rho = 3p$. The model was valid for $t \to -\infty$ to $t \to \infty$ having all physical and geometrical invariants finite and regular for the whole of spacetime. This was the first instance of a singularity-free cosmological model, satisfying all the energy and causality conditions and remaining true to general relativity (GR). Subsequently a family of singularity-free models has been identified[2]. In this communication we wish to point out that a simple and natural inhomogenization and anisotropization, appropriate for cylindrical symmetry, of the Friedman–Robertson–Walker (FRW) model with negative curvature leads to the same singularity-free family. It consists of the complete set of singularity-free general solutions of Einstein's equations for perfect fluid when cylindrically symmetric metric potentials are assumed to be separable functions of radial and time coordinates.

THE standard Friedmann–Robertson–Walker (FRW) cosmological model has been quite successful in describing the present state of the Universe. It prescribes a homogeneous and isotropic distribution for its matter content. It is though realized that homogeneous and isotropic character of spacetime cannot be sustained at all scales, particularly for very early times. Furthermore, not to have to assume very special initial conditions as well as for formation of large scale structures in the Universe it is imperative to consider inhomogeneity and anisotropy.

The first step in this direction came in the form of the study of anisotropic Bianchi models. Then inhomogeneity was also brought in and some inhomogeneous models were considered[3–7]. One of the main characteristics of the Einsteinian cosmology is the prediction of a big-bang singularity in the finite past. All these models (Bianchi as well as inhomogeneous) suffer from singularity at $t = 0$. This experience was strongly aided by the general result that under physically reasonable conditions of positivity of energy, causality and regularity etc., the initial singularity is inescapable in cosmology so long as we adhere to Einstein's equations (singularity theorems[8]). This gave rise to the folklore that the big-bang singularity is the essential property of the Einsteinian cosmology and it can only be avoided by invoking quantum effects and/or modifying Einstein's theory.

On this background it was really refreshing when Senovilla[1] obtained a new class of exact solutions of Einstein's equations without the big-bang singularity. It represented a cylindrically symmetric Universe filled with perfect fluid ($\rho = 3p$). It was smooth and regular everywhere, satisfied the energy and causality conditions and all the physical as well as geometrical invariants remained finite and regular for whole of spacetime. This marked the advent of singularity-free cosmology. It is important to recognize that the occurrence of singularity is not the general feature of Einstein's equations and for its avoidance it is not always necessary to resort to quantum effects and other fields. The classical Einstein's theory does admit cosmological models without singularity with physically acceptable behaviour for its matter content. It should be noted that prior attempts to construct singularity-free models had either to ascribe physically unacceptable behaviour for matter leading to violation of energy and causality conditions or to invoke quantum effects or modification of general relativity (GR)[9,10]. Senovilla's[1] was the first singularity-free solution true to GR, conforming to energy and causality conditions. Not only are physical parameters finite and regular, the solution has been shown to be geodesically complete[11]. Physically it means that a test particle will never encounter a singular state for arbitrarily large values of its affine parameter. That is the particle trajectory will never terminate anywhere.

One may wonder – how do these solutions escape singularity theorems[8]? It is because they do not satisfy one of the assumptions of the theorems, namely existence of compact trapped surfaces[8]. This assumption has always been a suspect and does not appear as obvious and natural as the energy and causality conditions. Violation of this means that nowhere in spacetime gravity becomes strong enough to focus fluid congruences in a small compact region so that all particles including photons get trapped. The occurrence of such a situation appears natural for gravitational collapse but by no means so for cosmology. For instance, even the open FRW model, that has big-bang singularity, never encounters a trapped surface. Hence existence of trapped surfaces cannot be a natural property for cosmological models. All previous attempts to construct singularity-free models have tampered the energy or causality conditions or GR. The remarkable feature of these models is that they adhere to all physically acceptable conditions and avoid the application of singularity theorems through nonexistence of trapped surfaces.

Ruiz and Senovilla[2] have separated out a fairly large class of singularity-free models through a comprehensive study of a general cylindrically symmetric metric with separable functions of r and t. Here we wish to establish a link between the FRW model and the singularity-free family by deducing the latter from the former. It works like this: transform the FRW metric with negative curvature into cylindrical coordinates and then introduce inhomogeneity and anisotropy by pasting the functions, that occur in FRW, with different powers in the metric coefficients. It is a simple and natural inhomogenization and anisotropization process that leads to the singularity-free family.

We begin with the FRW metric for the open Universe,

$$ds^2 = dt^2 - T^2(t)\left(\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \quad (1)$$

and transform it into cylindrical coordinates

$$ds^2 = dt^2 - T^2(t)\left(\frac{d\bar{r}^2}{1+\bar{r}^2} + (1+\bar{r}^2)dz^2 + \bar{r}^2 d\phi^2\right) \quad (2)$$

by the transformation

$$r = (\sinh^2 z + \bar{r}^2\cosh^2 z)^{1/2}, \ \tan\theta = \frac{\bar{r}}{\sinh z\sqrt{1+\bar{r}^2}}. \quad (3)$$

Further writing $m\bar{r} = \sinh(m\hat{r})$ and then dropping caps to write

$$ds^2 = dt^2 - T^2(t)(dr^2 + \cosh^2(mr)dz^2 + m^{-2}\sinh^2(mr)d\phi^2). \quad (4)$$

Let us now inhomogenize and anisotropize the FRW metric by writing

$$ds^2 = T^{2\alpha}\cosh^{2a}(mr)(dt^2 - dr^2) - T^{2\beta}\cosh^{2b}(mr)dz^2 - m^{-2}\sinh^2(mr)T^{2\gamma}\cosh^{2c}(mr)d\phi^2 \quad (5)$$

where we have used the coordinate freedom to write $g_{tt} = |g_{rr}|$. We could have as well used the form (2). Taking the natural velocity field $\mathbf{u} = T^\alpha\cosh^a(mr)dt$, the isotropy of fluid uniquely determines

$$T = \cosh(kt), \quad \alpha = \gamma. \quad (6)$$

With this the metric (5) is the family of singularity-free models identified by Ruiz and Senovilla[2].

Notice that $m^{-2}\sinh^2(mr)$ is simply to ensure 2π periodicity for the angle ϕ and elementary flatness near the axis and hence it does not participate in the inhomogenization and anisotropization process. Ruiz and Senovilla[2] have taken $\beta+\gamma=1$ and different undetermined functions of r in place of $\cosh(mr)$ and have found that all functions are expressible as powers of the same function $\cosh(mr)$. For time dependence $T = \cosh(kt)$ is the general solution. Even if we take $\beta+\gamma \neq 1$ and different $T(t)$ functions, it turns out that they can all be given as the powers of the single function, as given by (6). Thus the metric (5) with (6) forms the complete set of singularity-free solutions of cylindrically symmetric metric with separable functions of r and t.

For singularity-free models, both Weyl and Ricci curvatures should be regular and their regularity for the metric (5) demands $\alpha = \gamma$. The isotropy of pressure constrains the parameters and it can be shown that the only two following cases give rise to singularity-free models:

(i) $b = c, \alpha = \gamma, \alpha + \beta = 1, a = -b/(1+2b), k = (1+2b)m$,

(ii) $b + c = 1; \alpha = \gamma, \alpha + \beta = 1, a = -b(1-b), k = 2m$.

In the former case there does not occur an equation of state $\rho = \mu p$ in general, however for $b = -\frac{1}{3}$ we obtain the Senovilla[1] radiation model with $\rho = 3p$. In the latter case it is always $\rho = p$ giving the stiff matter model[12]. The matter-free limit ($\rho = 0$) of the stiff matter model yields two distinct singularity-free vacuum solutions. It may be noted that all these are the general solutions in the given setting.

The kinematic parameters, expansion, shear and acceleration are given by

$$\theta = (\alpha+1)k\sinh(kt)\cosh^{-\alpha-1}(kt)\cosh^{-a}(mr)$$

$$\sigma^2 = \tfrac{2}{3}(2\alpha-1)^2 k^2\sinh^2(kt)\cosh^{-2(\alpha+1)}(kt)\cosh^{-2a}(mr)$$

$$\dot{u}_r = -am\sinh(mr)\cosh^{-a-1}(mr)\cosh^{-\alpha}(kt).$$

It is clear that the above kinematic parameters are regular and finite all through the spacetime. We have verified that so are the physical parameters ρ, p and the Weyl curvatures. The general behaviour of the model is the same as that of the Senovilla's radiation model. As $t \to \pm\infty$ density and curvatures tend to zero though the metric does not go over to the Minkowski form. The Universe begins with low density at $t \to -\infty$, contracts to a dense state at $t = 0$ (where density can be made as large as one pleases by specifying the parameter k) and then starts expanding to reach the initial state ($t \to -\infty$) as $t \to \infty$. At $t = 0$ both expansion and shear change their sense.

It appears that presence of shear and acceleration seems to play an important role in avoidance of singularity. It is conceivable that they do not let fluid congruences to focus into small enough a region to form trapped surfaces leading to singularity. When fluid congruence has non-zero acceleration, there occurs spatial *pressure gradient* which will counteract gravitational attraction to give rise to a bounce to the model. This is how contraction changes into expansion at $t = 0$ without letting the Universe to pass through a singular state. The shearing of the congruence has a defocusing effect. Their presence alone however is not sufficient to avoid singularity as there exist singular models with non-vanishing shear and acceleration. For instance replace $\cosh(kt)$, in the above by $\sinh(kt)$ to get a class of models with the big-bang singularity with shear and acceleration present. Thus it may perhaps be the necessary condition but not sufficient. This added with the regularity of Weyl and Ricci curvatures, and energy conditions may lead to sufficiency. We have also *verified that the metric (5) is geodesically complete*[13] for $\alpha \geq 0$, $\alpha+\beta \geq 0$, $\alpha \geq \beta$, $a \geq 0$, $a \geq b$, $a+b \geq 0$, and $b \leq 0$.

The metric (5) with (6) can as well be cast in the form

$$ds^2 = (1+k^2t^2)^{a-1}(1+m^2r^2)^a \, dt^2$$

$$-(1+k^2t^2)^{\alpha}(1+m^2r^2)^{a-1} dr^2 - (1+k^2t^2)^{\beta}$$

$$(1+m^2r^2)^b dz^2 - r^2(1+k^2t^2)^{\alpha}(1+m^2r^2)^c d\phi^2 \quad (7)$$

which reduces to the FRW from (2) for $\alpha = \beta = 1$, $a = c = 0$ and $b = 1$, where $T(t) = (1 + k^2t^2)$. It is interesting that if one just uses hyperbolic or $(1 + k^2t^2)$ functions, which are clearly the obvious choice for singularity-free spacetime, one ends up with the family of singularity-free models. However the cylindrical symmetry seems to play an important role. The above prescription does not obviously work in spherical symmetry. A tentative prescription for a singularity-free spacetime may be as follows: Use non-singular elementary functions having no zeros as metric potentials, and ensure regularity of Weyl and Ricci curvatures with non-zero shear and acceleration. This is just the broad setting, the fluid consistency conditions and energy conditions are then to be satisfied. This prescription does yield acceptable fluid models for cylindrical symmetry.

One may ask the question, how robust is the singularity-free framework in relation to accommodating other force fields? It turns out that viscosity cannot be included without sacrificing positivity of viscosity coefficients for all time[14] while the radial heat flow can *easily be included*[15]. *Both the cases above can be generalized* to have radial heat flow. Note that for $\rho = \mu p$, μ can have the only two discrete values ($\mu = 1, 3$). If we introduce massless scalar field along with perfect fluid in the case (i), the resulting fluid can have an equation of state, $4 > \mu > 3$, *opening out a narrow window for* μ.

Finally, the most pertinent question for the singularity-free models is: how to evolve them into the (present day) FRW models? The question is inherently very difficult because the former are cylindrically *symmetric wherein a direction is singled out while the* latter are spherical having no identifiable direction. It may be noted that the ratio of shear to expansion, that measures the anisotropy, is a constant for these models. We have here established a linkage between the two, which may help in reconciling them together. The affirmative answer to this question will have very important bearing on our overall cosmological perception of the Universe and, in particular, for the early Universe cosmology. The other question is, do there exist general solutions without any symmetry or is cylindrical symmetry singled out for singularity-free solutions? Details will be given in the forthcoming paper[16].

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Note added in the proof. Since the submission of the paper we have proved the general result establishing the uniqueness *of the metric (5) with (6) for singularity-free fluid* models when space-time metric is separable in space and time coordinates.

Use of organic geochemical markers in elucidating the origin of salinity in coastal groundwaters

V. N. Varma and B. S. Sukhija

National Geophysical Research Institute, Hyderabad 500 007, India

The present paper demonstrates the possible application of organic geochemical markers in elucidating the origin of salinity in coastal groundwaters. The study carried out in the ground waters, estuarine sediments and aquifer material in the coastal Karaikal region of Cauvery delta has shown that the *estuaries/modern marine bodies and their intrusions* in groundwaters can be evidenced by the presence of *hopanoic and vaccenic acids whereas the palaeomarine* intrusions in groundwater have the signatures of biomarkers like those of palmitoleic and oleic acids. This finding is supported by iodide/chloride ratio and radiocarbon dating of groundwaters.

THE study of salinity in groundwaters has gained importance as being one of the environmental problems. Moreover, the salinity of groundwater has direct bearing on agriculture. The salinity in the groundwater could be due to a number of natural