Riemann surfaces, projective curves and the iso-spectral problem

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1. Historical introduction

I should say at the outset that this is not meant to be a complete history of the theory of Riemann surfaces and is a patchwork of my knowledge of the development—hence it is occasionally historically incorrect and sometimes even anachronistic. Naturally it reflects my own biases.

In a sense the theory of infinite series is the predecessor of the theory of Riemann surfaces. Infinite series were extensively studied by Euler. He showed how every 'interesting' function could be *calculated* using infinite series and all interesting operations like integration, differentiation and so on can be carried out using series.

The theory of infinite series had its logical (but in no sense obvious!) development into the theory of functions of one complex variable under Riemann, Cauchy and later Weierstrass. Instead of series being a means to compute functions, it turned out that if one was concerned with complex valued functions of one complex variable, then infinite series and functions were 'locally' interchangeable notions. From this 'local' starting point Riemann was able to develop the corresponding 'global' notion—the concept of a Riemann surface. This in turn raised numerous problems in the theory of functions of one complex variable—with geometry being the predominant theme. To solve many of these Riemann introduced (in analogy with the complex torus associated with *elliptic functions* studied by Abel and Jacobi) a certain *torus* naturally associated with a *compact* Riemann surface, nowadays referred to as its Jacobian. A vital tool in the study of this torus and the associated Riemann surface was Riemann's *theta function*, a function of several complex variables. The *Riemann singularity theorem* showed how the singularities of the zero locus of this function yielded results in function theory.

Much of Riemann's theory was left incomplete due to his early death (and possibly due to a profusion of ideas far too numerous for him to completely elucidate for his contemporaries). It was left to Weierstrass, Poincaré, Klein and Weyl to fill out the outlines of the function theoretic aspect. At this point we see a branching of the theory of Riemann surfaces into two streams. From the point of view of function theory, compact Riemann surfaces appeared to be completely understood and this led Teichmüller, Ahlfors and others to devote their time to non-compact surfaces. We shall not follow this interesting stream of thought.

Through Riemann's work it was clear that if the Riemann surface associated with a function f is compact, then f is an algebraic function of z. Hence we may associate a projective (algebraic) curve to a Riemann surface. Geometers like Max Noether, Schubert and others had built up an impressive amount of theory from this geometric point of view. The work of Baker, Emil Artin, Emmy Noether (daughter of Max) and others brought a distinctly algebraic flavour to this subject. This algebraic aspect was more fully laid out by Zariski and Chevalley. Along with many others (Weil and Chow being among the prominent) they put together the foundations of *algebraic geometry*. Almost before this work was complete however, Alexander Grothendieck re-wrote their foundations and built upon these foundations far-reaching generalizations of many of their ideas. Following Grothendieck's approach, Mumford completed the geometrization of the entire theory of Riemann surfaces, including the Jacobian and the theta function so that the whole theory was now workable for curves over any field (or ring!). The succeeding work of Kempf, Griffiths, Harris and others made it appear that *all* outstanding geometric problems from the theory of curves had been solved.

Since the sixties there had been a group of mathematicians in India—C. S. Seshadri, M. S. Narasimhan, S. Ramanan and others—who had begun (with a paper of Weil) to study *vector bundles* on Riemann surfaces to generalize the notion of the Jacobian. Already in the seventies the work of Krichever (which was partly the rediscovery of some work of Baker and his students!) and later Drinfeld had shown how the study of vector bundles on curves led to applications in other areas of mathematics (like the solutions of certain partial differential equations). In a profusion of papers in the eighties the theoretical and mathematical physicists (notably string theorists) have shown a great interest in these problems; firstly for the theory of *solitons* and later the questions about *moduli*. Combining the work of the analysts and the geometers, Wolpert, Harer, Miller and others could prove many interesting properties about the moduli space of Riemann surfaces. This work had a combinatorial flavour which has been further expanded in the work of the string theorists.

So in some sense we have reached a third phase in the theory of Riemann surfaces where the function theory and geometry of Riemann surfaces are so well understood that one can begin to study the moduli space from a computational and combinatorial point of view. This is not to say that no problems in the analytic and geometric aspects remain—but that many of the interesting problems about Riemann surfaces are now of a combinatorial nature.

2. Solitons and iso-spectral deformations

The theory of Riemann surfaces was of some interest to theoretical physicists of the previous century when it was used as a 'toy' model on which electromagnetic field theory was studied. This was later followed up by Klein who used the physics as a justification for the application of the Dirichlet principle (the proof then in existence had some gaps). More recently, the theory has been applied in a different way to various problems in physics and engineering. The particular class of problems considered here can be classified as problems of *iso-spectral deformation*.

Given an operator P (say a linear operator on a function space) we can ask for all deformations $P(x_1,\ldots,x_n)$ of this operator that preserve the eigenvalues *along with their multiplicities*, i.e. deformations that preserve the *spectrum*. Such a deformation is called an *iso-spectral deformation*. One way to formalize this notion is as follows. We consider the largest *commutative* algebra $\mathscr{A}$ of operators that contains P. It is well known that the spectrum of P determines the structure of $\mathscr{A}$ and vice-versa. Then for each $P(x_1,\ldots,x_n)$ we have an algebra $\mathscr{A}(x_1,\ldots,x_n)$. The problem is to choose deformations such that $\mathscr{A}\cong\mathscr{A}(x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$. It was shown by Lax that we need only consider the special case in which the deformation is contained in the *adjoint orbit*, i.e. there should be operators B_k such that

$$\frac{\partial P}{\partial x_k}=[B_k,P],$$

where [,] denotes the commutator of two operators and the differentiation is being performed on the coefficients of the operator $P(x_1,\ldots,x_n)$. Moreover, we have the natural condition for integrability

$$\frac{\partial B_k}{\partial x_l}-\frac{\partial B_l}{\partial x_k}=[B_l,B_k].$$

This way of stating the isospectral problem is called the Zakharov-Shabat formulation.

2.1 *Lax operators*

The situation where P is a monic ordinary differential operator will lead us to Riemann surfaces. We first enlarge the scope of the problem by allowing *pseudo-differential operators.*

Let $\mathcal{O}$ be the space of 'functions' on which our operators operate. Let ∂ denote the given differentiation on $\mathcal{O}$ so that any ordinary differential operator is of the form

$$Q=a_n\partial^n+a_{n-1}\partial^{n-1}+\cdots+a_0,$$

where the a_k are 'functions' (elements of $\mathcal{O}$). A pseudo-differential operator is a *formal expression* of the above type where we allow *infinitely many terms* with *negative* powers of ∂.

$$S=a_n\partial+\cdots+a_0+a_{-1}\partial^{-1}+\cdots$$

The Leibniz rule is extended in a natural way by defining

$$\partial^{-1}a=a\partial^{-1}-\partial(a)\partial^{-2}+\partial(\partial(a))\partial^{-3}-\cdots.$$

This gives us the algebra of all pseudo-differential operators. One of the advantages of working in this algebra is that any monic operator (see below) can be inverted. Though these operators are purely formal, the final results we obtain will actually give *convergent* solutions.

Let P be a monic ordinary differential operator, i.e.

$$P=\partial^n+a_{n-1}\partial^{n-1}+\cdots+a_0.$$

Let us assume that $g=\exp(a_{n-1}/n)$ is an element of $\mathcal{O}$, i.e. we assume that the equation $(\partial+a_{n-1}/n)g=0$ has a solution in $\mathcal{O}$. Then replacing P by $g^{-1}Pg$ we have the simplified operator

$$P=\partial^n+0\partial^{n-1}+a_{n-2}\partial^{n-2}+\cdots+a_0.$$

We can now find a pseudo-differential operator

$$L=\partial+0+u_1\partial^{-1}+\cdots$$

such that $L^n=P$. This requires us to solve a succession of *linear* equations for u_k which can be solved inductively. One can easily see that the deformation theory for P yields the following evolution equations for L,

$$\frac{\partial L}{\partial x_k}=[B_k,L].$$

Thus the study of P can be replaced by the study of operators like L. These are called *Lax operators.*

Example. An example of some interest to theoretical physicists as a 'toy' model is the Schrödinger operator $P=\partial^2+u_0$.

2.2 *Commutative algebras of operators*

In order to study the spectrum of our monic ordinary differential operator P we can study the maximal commutative algebra $\mathscr{A}$ of ordinary differential operators such that P lies in it.

Let k be the collection of all 'constant functions', i.e. those a in $\mathscr{O}$ such that $\partial(a)=0$. Let $f(T)$ be a Laurent power series

$$f(T)=\alpha_n T^{-n}+\alpha_{n-1} T^{-n+1}+\cdots+\alpha_0+\alpha_{-1} T+\cdots$$

where α_k are 'constants'. We can show that the collection of all pseudo-differential operators that commute with L is the algebra of all operators of the form $f(L^{-1})$. In other words this algebra is *isomorphic* to the algebra $k((T))$ of formal Laurent series in the variable T. Any such operator has two parts $f(L^{-1})=f(L^{-1})_+ + f(L^{-1})_-$ consisting of the ordinary differential and the pseudo-differential operator parts. The commuting algebra $\mathscr{A}$ is then the collection of operators $f(L^{-1})=f(L^{-1})_+$.

In particular, one can see that $\mathscr{A}$ yields a subalgebra A of $k((T))$ with the following properties:

1. There are no elements of A that have the form

   $f(T)=\alpha_1 T+\alpha_2 T^2+\cdots$.

2. There is some power of T^{-1} which lies in A.

As a first step we can try to compute all subalgebras A of $k((T))$ with this property. We will show next how these algebras lead us to Riemann surfaces. After that we will return to the iso-spectral problem and show how it can be solved using the geometry of Riemann surfaces.

3. Riemann surfaces and algebraic curves

One way of obtaining Laurent series is to study holomorphic functions on the complex plane minus the origin. The classical study of such a function begins with the study of its *domain* of definition. This yields a Riemann surface. In some cases this Riemann surface can naturally be thought of as a projective algebraic curve. We will identify all algebras A as in section 2 with the algebras of certain functions on such curves.

3.1 *Functions of one complex variable*

Let $f:D\to\mathbb{C}$ be a complex valued function on the unit disk $D=\{z:|z|<1\}$ inside the complex plane. If the function is holomorphic, i.e. if it satisfies the Cauchy–Riemann condition at every point of D, then we have a series expression:

$$f(z)=\sum_{n=0} a_n z^n.$$

Conversely, given such a series there is a well known condition — the Weierstrass M-test — that will ensure that f is a well-determined holomorphic function in the unit disk. Hence from here onwards we do not distinguish between holomorphic functions on a disk and the corresponding convergent series. Of course, the series stops converging at some point (possibly infinity!) but Riemann invented the beautiful notion of *analytic continuation* which circumvented this. (The idea behind analytic continuation was vastly generalized by Leray in the fifties by introducing the concept of *sheaves*. This was developed in the fifties by Kodaira, Serre and Leray. The new foundations of Grothendieck were based on this important notion.)

It may happen that there is another disk D' in the complex plane which meets D, e.g. $D'=\{z:|z-1|<a\}$, and a holomorphic function $g:D'\to\mathbb{C}$ such that $f(z)=g(z)$ for all z in $D\cap D'$. Since any holomorphic function on D is determined by its value on an open set like $D\cap D'$, we may consider the function f as being *continued* (to be defined) to D' using g'. This raises the natural question of the largest region or *domain* of definition of the function f.

Example. Let us take the function on the unit disk $f(z)=(1+z)^{1/2}/(1-z)$. We may choose any disk D' which does not contain $\{1,-1\}$ and show that there is a way of continuing this function. But let us take the sequence of disks arranged around the point -1 in the complex plane:

$$D_0=\{z:|z|<1\}$$
$$D_1=\{z:|z-(1+i)|<1\}$$
$$D_2=\{z:|z-2|<1\}$$
$$D_3=\{z:|z-(1-i)|<1\}$$
$$D_4=\{z:|z|<1\} \text{ again!}$$

We let $f_0=f$ and let f_i be the function on D_i obtained by continuing the function f_{i-1} defined inductively on D_{i-1}. We find that $f_4(z)=-(1+z)^{1/2}/(1-z)$.

As the above example shows, we must keep track of the *function* g obtained by continuation in our definition of the domain. This domain is obtained by putting together all pairs (g,D') consisting of a disk D' in the complex plane and a holomorphic function g on it, such that this pair is linked to the original pair (f,D) as follows. There is a sequence of disks $D_0, D_1,\ldots,D_n$ which are interlinked, i.e. D_r meets D_{r+1}, and functions f_r on D_r which agree on the overlaps; moreover, we begin with $D_0=D$ and $f_0=f$ and end with $D_n=D'$ and $f_n=g$. Of course we may subsume (g,D') within (h,D'') if D' is contained in D'' and h restricts to g on D'. What we obtain is called the *Riemann surface* associated with the function f.

We see that the same disk may appear in our collections many times. This is usually expressed by saying that the Riemann surface of f consists of many *sheets* lying over the complex plane (which can be thought of as the Riemann surface of the *identity* function z). Each of the components (g, D) is referred to as a *branch* of the Riemann surface.

In the above example, the function is well determined in a small enough disk around any point other than 1 and -1. In the general case also, there may be many points of the plane over which some *branch* of the function cannot be extended. Such points are referred to as *singularities* of the function. We now study the function in a neighbourhood of a singularity.

We assume that the singularity is *isolated*, i.e. we have a function f which can be continued over every point of the punctured unit disk

$$D^* = \{z : 0 < |z| < 1\}.$$

From the usual theory of analytic functions we know that such functions fall into three classes as follows.

1. The function f has an expression of the form

$$f(z) = \sum_{i=-N}^{\infty} a_i z^i,$$

i.e. a *Laurent series* expansion exists. In this case we say the function has a *pole* of order N at the origin and define its 'value' to be ∞. This makes sense since $1/f$ is actually analytic on the entire disk and takes the value 0 at the origin.

2. The function is multivalued on the disk and has the form

$$f(z) = \sum_{i=-N}^{\infty} a_i u(z)^i,$$

where $z = \exp(u(z))$. In other words, we have a function of $\log(z)$. This is the case of a *logarithmic singularity.*

3. One of the special cases of the above case is when f is actually a function of $w = \exp(u(z)/n)$ for some integer n. In this case f is n-valued over the disk and is a function of $w = z^{1/n}$. We then say the function (or more correctly the corresponding Riemann surface) is *ramified of order n* over the origin.

4. The function has an *essential singularity* at the origin.

We restrict our attention to the 'finite' cases (1) and (3) above. If *all singularities of all branches* of our function are of these two types then Riemann showed that we can 'adjoin' some more points to the domain of definition and obtain a *compact Riemann surface.*

One way of constructing such functions f is as follows. We take a polynomial function

$$P(z, T) = a_0(z) T^n + a_1(z) T^{n-1} + \cdots + a_n(z),$$

where $a_k(z)$ are polynomial functions of z. Let f be a function which satisfies this polynomial equation, i.e. $P(z, f(z))$ is zero for all z in D; we call such an f an *algebraic function* of z. The function in the above example is one such. The crucial point is *any* function f which has no logarithmic or essential singularities at any branch over the Riemann sphere is of this type (and the Riemann surface is then compact).

Example. The function considered in the previous example yields the equation

$$P(z, T) = (1-z)^2 y^2 - (1+z)$$

which is clearly such that $P(z, f(z)) = 0$.

Conversely, consider the associated *plane curve* consisting of pairs (x, y) such that $P(x, y)$ is zero. Since we have been looking at complex valued functions of a complex variable we may look for solutions where x and y are complex numbers (in fact since we also allow ∞ as a value we can look for solutions in the complex projective plane!). Now we can see that outside a finite set of choices $x = a$ in $\mathbb{C}$, the choice of one root y for $P(a, T)$ as a polynomial in T yields a choice of root $y(z)$ for each z in a disk $\{z : |z-a| < b\}$ around a. Moreover this function $y(z)$ is a holomorphic function of z, which is a branch of the function f which was chosen in the previous paragraph.

Example. In the example considered above, the Riemann surface is associated with the plane algebraic curve defined as the locus of points $(X:Y:Z)$ of the complex projective plane where the homogeneous polynomial

$$P(X, Y, Z) = (Z-X)^2 Y^2 - (Z+X) Z^3$$

vanishes. There is a point $o = (0, 1, 1)$ in this locus where the partial derivatives of P do not all vanish. In other words o is a *smooth point* of the curve.

Thus we have built a 'picture' for our Riemann surface. Outside a finite set of points it is the set of (complex) points of a plane algebraic curve; moreover, the coordinate function z is obtained by projection to the x-axis and the function f, by projection to the y-axis.

3.2 *Projective algebraic curves*

We saw that to every compact Riemann surface there is associated a plane algebraic curve. In fact one may show that a compact Riemann surface can be *embedded* in complex projective n-space $\mathbb{P}^n_{\mathbb{C}}$. Thus the study of compact Riemann surfaces is also the study of smooth complex projective curves (all such are in fact *algebraic curves*, i.e. defined by polynomial equations).

The field of meromorphic functions on an algebraic curve is a *function field* of transcendence degree one over $\mathbb{C}$; i.e. the field is generated over $\mathbb{C}$ by two elements that satisfy exactly one polynomial relation. For example, if we have a plane algebraic curve defined as the zero locus of a polynomial equation $P(X,Y)=0$, then the field of meromorphic functions on the curve is generated by two elements x and y which are subject to the relation $P(x,y)=0$.

Let o be any point of our plane algebraic curve. We choose a local parametrization $(x(T), y(T))$ near this point. Using this parametrization any meromorphic function on the curve can be written as a Laurent series in T. The meromorphic functions that have poles *only at o* give us a subalgebra A of the algebra $k((T))$ of all Laurent series. Any elements of A that do not even have a pole at o, i.e. which have no poles at all, must be constants by Louiville's theorem. In particular, there are no elements of A that have only positive powers of T. Thus we have an algebra as required in section 2.

Example: For the curve given by $(z-x)^2y^2-(z+x)z^3=0$ we write the parametrization $x(T)=T$, $y(T)=(1+T)^{1/2}/(1-T)$ and $z=1$ is a neighbourhood of the point $o=(0,1,1)$. One shows that the algebra A is finitely generated.

Any subalgebra A of $k((T))$ which has no elements with only positive powers of T, and which contains some power of T^{-1} (as in section 2) satisfies

1. There is a positive integer r (called the *rank*) such that for any sufficiently large integer n, there is an element of A of the form

$$f(T)=T^{-nr}+a_{-nr+1}T^{-nr+1}+\cdots.$$

2. The algebra A is finitely generated.

As a consequence it follows that there is a projective algebraic curve X and a point o on it such that X is smooth near o and A the algebra of all meromorphic functions on X which have poles only at o.

4. Solution of the iso-spectral problem

We now examine the combined results of the previous two sections. For a monic ordinary differential operator P we introduced the maximal commutative algebra $\mathscr{A}$ containing it. Later we showed that such an algebra is isomorphic to a subalgebra A of the algebra of the Laurent power series of a certain kind. We then showed how such an algebra corresponds to a projective algebraic curve X and smooth point o on it. In terms of spectral theory these results mean that each point of X except o corresponds to some simultaneous eigenfunction for all elements of $\mathscr{A}$. The corresponding eigenvalue being just the value of this element considered as a function on X.

To each point p of X we associate also the space E_p of simultaneous eigenfunctions corresponding to it. In case E_p is a vector space whose rank is independent of the point p we get a *vector bundle* over the Riemann surface, at least outside o. We may extend this bundle in many ways to a bundle over all of X. These many ways are what give rise to the deformations of the operator P.

In section 4.2 we will see how these deformations can be constructed using the geometry of projective algebraic curves. For simplicity we assume that we are in the case where X has no singularities. We also assume that the rank of the eigenspaces above is 1.

4.1 *Some more geometry of projective curves*

Given any non-zero function f on a curve we can associate to it its *zeroes* and *poles*; these form a *finite* set of points on the curve. To any zero of f we can associate its *order* (of zero) which is computed by making a local expansion of the function in terms of a local coordinate as in section 1. Similarly, the poles of f are the zeroes of $1/f$ and so also have an associated *order* (of pole). We adopt the convention that a pole of order n of f is a zero of order $-n$. The *divisor* of f is then the collection of pairs (p,a_p), where p is a point of the curve and a_p is the order of zero of f at p. We refer to *any* such collection $D=\{(p,a_p)\}$, where a_p are integers *only finitely many of which are non-zero* as a *divisor*. The sum of all the integers a_p (this is a finite sum) is called the *degree* of the divisor. If the divisor is associated with a function it is of degree 0.

We note that if f and g are functions on the curve and have associated divisors $\{(p,a_p)\}$ and $\{(p,b_p)\}$, then fg has the divisor $\{(p,a_p+b_p)\}$. As we saw above the divisor associated with $1/f$ is $\{(p,-a_p)\}$. The formula $\{(p,a_p)\}+\{(p,b_p)\}=\{(p,a_p+b_p)\}$ defines the structure of an additive group on the set of divisors. Hence we refer to it as the *group of divisors* on our curve. Moreover we have just seen that there is a natural *group homomorphism* from the *multiplicative group* of non-zero functions on our curve to its group of divisors. The *divisor class group* is the quotient of the group of divisors by the image of this homomorphism.

Just as we can talk about meromorphic functions on the curve we can talk about meromorphic differentials. These are of the form $f\,dg$, where f and g are meromorphic functions. If we express g in local coordinates at p as $g(z)=\sum_{i=0}^{\infty}a_iz^i$, then dg vanishes at p if and only if

$$\frac{dg}{dz}\Big|_p=a_1=0.$$

The poles of order n of g will give rise to poles of order $n+1$ of dg. Thus we have a divisor K associated with dg. Now for any other function g' on the curve (dg'/dg) is a function and so we see that the image of K in the divisor class group of the curve is well-determined. This divisor class is also denoted by K and is called the *canonical divisor* (class) of the curve. The degree of the divisor K is an important invariant of the curve. It is even and the number $\gamma=(\deg(K)/2+1)$ is called the *genus* of the curve.

Given any divisor $D=\{(p,a_p)\}$ we say D is *effective*, if $a_p \geq 0$ for all p. Now for any divisor D we can look for all functions f such that $D+\operatorname{div}(f)$ is effective. The set of such functions is a vector space $l(D)$ of finite dimension. Up to scalar multiples this is also the collection of all effective divisors which have the same image as D in the divisor class group.

The *Riemann inequality* gives a lower bound on the dimension of $l(D)$,

$$\dim l(D) \geq \deg(D)+1-\gamma,$$

where γ is the genus of the curve. This inequality obtained by Riemann was improved by Roch to get the Riemann–Roch identity.

$$\dim l(D)-\dim l(K-D)=\deg(D)+1-\gamma.$$

The latter is a very useful tool in the study of the projective geometry of the Riemann surface. An easy exercise is to use it to show that if $\deg(D)>\deg(K)$, then the Riemann inequality becomes an equality.

The final object we wish to consider is the Jacobian of the curve. Classically, this is a torus constructed by computing the periods of integrals on the Riemann surface. Let $\omega_1,\ldots,\omega_\gamma$ be a basis of $l(K)$ which we are identifying with the space of differential forms with no poles on X. For any closed loop C on X we can compute the vector of integrals

$$\left(\int_C \omega_1,\ldots, \int_C \omega_\gamma\right).$$

The collection of all such vectors gives us a complete *lattice* Λ in the complex linear space $\mathbb{C}^\gamma$.

The Jacobian torus is in a natural way identified with the quotient $J(X)=\mathbb{C}^\gamma/\Lambda$. Riemann (and later S. Lefschetz) showed that $J(X)$ can also be embedded in complex projective n-space for some n. For any pair of points p and q on X, we can choose a path P joining these two. The vector of integrals (studied by Abel and called *Abelian integrals*)

$$\left(\int_P \omega_1,\ldots, \int_P \omega_\gamma\right)$$

depends on P. If we choose another path P' joining the same two points then we have a closed C loop on X obtained by following P in the forward direction and then P' in the reverse direction. We then have an equation

$$\left(\int_P \omega_1,\ldots, \int_P \omega_\gamma\right)-\left(\int_{P'} \omega_1,\ldots,\int_{P'} \omega_\gamma\right)=$$
$$\left(\int_C \omega_1,\ldots,\int_C \omega_\gamma\right).$$

Hence, we have a well-defined point of $\mathbb{C}^\gamma/\Lambda$ associated with any *ordered* pair of points p and q on X. This procedure can be extended to associate a point $\eta(D)$ for each divisor D of degree 0 on X.

The map η is additive, i.e. $\eta(D+D')=\eta(D)+\eta(D')$. It was studied by Abel, Jacobi and later more extensively by Riemann and is called the *Abel–Jacobi map*. The divisors of functions are in its kernel. The problem considered by Abel and solved by Jacobi and Riemann is to show that any divisor in the kernel is indeed the divisor of a function, i.e. $\eta(D)=0$ *if and only if* there is a function f on X such that its divisor is D.

Thus we see that a detailed geometric study of η can tell us exactly what kinds of functions exist on the Riemann surface. This approach was investigated first by Riemann who formulated the *Riemann singularity theorem*. A detailed proof of this theorem was only obtained quite recently by Kempf.

As a result of the Riemann inequality, for a divisor D of degree $\leq (p-1)$ the space $l(D)$ has dimension ≥ 0. For a general divisor one may show that this dimension is actually zero. Following some (conjectural) work of Brill and M. Noether analysing the Abel–Jacobi map, Griffiths and Harris were able to obtain far more precise results saying when $l(D)$ has dimension bigger than 1 for $\deg(D) \leq (p-1)$. Numerous other problems of a geometric nature were also made accessible by their techniques.

An important technique called *degeneration* is used by Griffiths and Harris and has recently been developed further by Eisenbud and Harris. Loosely speaking this is a 'bend and break' technique. By the study of properties of *singular* ('broken') curves one can recover information about smooth ones. The analogues of the spaces $l(D)$ for singular curves are called *limit linear series*. The latter technique is so powerful that Eisenbud and Harris were able to give proofs of almost all outstanding geometric problems about curves by this method (including the mysterious appearance of the number 26 as the dimension of space-time).

4.2 *Deformations*

For any divisor D on X which has degree 0 we consider the space

$$E(D)=\bigcup_{n\geq 0} l(D+no).$$

By the Riemann inequality the latter spaces keep on growing in dimension and eventually, for $n \gg 0$ the space $l(D+no)$ has dimension n. We can also vary D by adding divisors D' of the following form. For some point p of X we can consider the divisor which has $a_p = -1$ and $a_o = 1$, $a_q = 0$ for all points X other than o and p. By doing this many times we have an infinite parameter variation of the data.

The space $E(D)$ can be identified with a subspace of $k((T))$. This lies in the class of subspaces V with the property that for all m sufficiently large there are exactly m linearly independent Laurent series in V of the form

$$f(T) = a_{-m} T^{-m} + a_{-m+1} T^{-m+1} + \cdots.$$

The collections of all such spaces forms the *infinite Grassmannian*. Replacing D by $D+D'$ gives another such subspace. As D' approaches the trivial divisor this gives an infinitesimal action. This infinitesimal action is the iso-spectral flow. We note here that in the case of rank 1 the integral manifold of this flow is the Jacobian of the curve X.

The infinite Grassmannian and the associated group action is a purely combinatorial construction, involving the symmetric functions and associated polynomials. Since all *pointed* Riemann surfaces occur on the infinite Grassmannian we can now attempt to study the collections of all curves in a purely combinatorial fashion.

5. Further reading and references

A much more complete historical sketch of the theory of Riemann surfaces (and algebraic geometry in general) may be found in the book: Shafarevich, I. R., *Basic Algebraic Geometry*, Springer–Verlag, Berlin, 1977. This book also contains more details about sections 3.2 and 4.1.

For the formalism of Lax operators (section 2.1) and their use in solving the iso-spectral problem (section 2.2) we have followed the exposition of: Motohico Mulase, *J. Diff. Geom.*, 1984, **19**, 403–430. Much of the formalism comes from the original papers of Burchnall and Chaundy.

The definition of Riemann surfaces arising out of one variable function theory (section 3.1) can be studied from chapter 8 of the book: Lars Ahlfors, *Complex Analysis*, McGraw-Hill Kogakusha, International student edition, 2nd edition, 1966.

The geometry of projective curves is a well-developed subject and the book (Arabarello, E., Cornalba, M., Griffiths, P. A. and Harris, J., *Grundlehr. Math. Wiss.*, 1985, 267) covers all the unproved assertions of section 4.1 and much more. Section 4.2 is adapted from a Hilbert space approach of Segal and Wilson: Wilson, G., in *Geometry Today*, Proceedings of Giornata di Geometria, Rome 1984, Birkhäuser, Boston, 1985.

The Weil conjectures

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1. Diophantine equations

In number theory, one of the problems of basic interest is to find all integer solutions of an equation

$$f(X_1, \ldots, X_n) = 0,$$

where f is a polynomial with integer coefficients; such problems are known as *Diophantine problems*. It is also of interest to study systems of such equations, and to consider algebraic integer solutions (for example, solutions with $X_i = \Sigma_j a_{ij} \zeta^j$, where ζ is a primitive nth root of unity, and the a_{ij} are integers).

As a first step, one may instead look for integers X_i such that $f(X_1, \ldots, X_n)$ is *divisible by a given prime number* p. Since 0 is divisible by p, this is certainly an 'easier' problem to solve, in the sense that if it has no solution, neither does the original Diophantine problem.

Equivalently, one considers $\mathbf{F}_p$, the *integers modulo* p; one description of these is as follows—$\mathbf{F}_p = \{0, 1, \ldots, p-1\}$, where we define the sum of two such numbers to be the remainder obtained on dividing their sum (as integers) by p. Their product is similarly defined as the remainder obtained on dividing the integer product by p. These modified operations produce a *field*, i.e. an algebraic system in which one can perform the usual operations of addition, multiplication and division by non-zero elements, and these operations have the standard properties. This is an example of a finite field. There is a mapping $\mathbf{Z} \to \mathbf{F}_p$ (called 'reduction modulo p') which associates to each integer the