Resolution of singularities in various characteristics

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1. What is a singularity?

Singularity of what? Say, of a curve or a surface. So, what is a curve? A *curve* is a wiggling line which may cross on itself and may also have some sharp corners or beak-like features. Such special points, where the curve crosses itself, see Figure 1, or has a beak, see Figure 2, is a *singularity*. The first type of singularity is called a *node* and the second type is called a *cusp*.

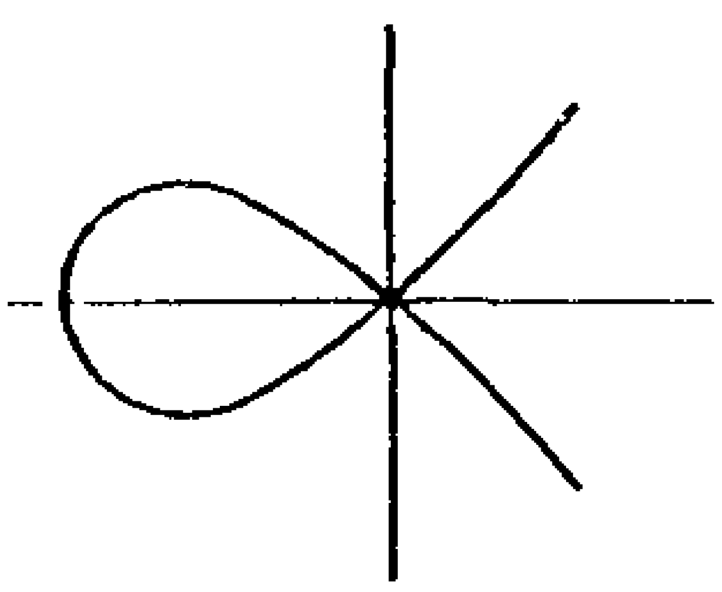

Figure 1.

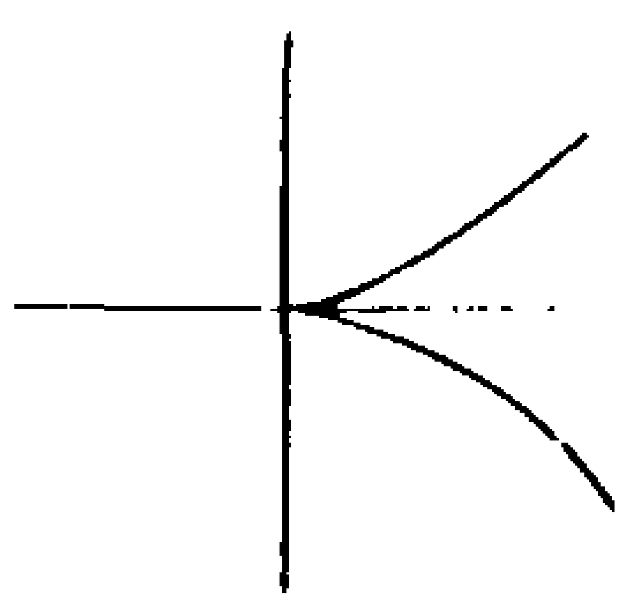

Figure 2.

Algebraically speaking, a curve can be given by an equation. For example a straight line through the origin has the equation $y=mx$, where m is the slope, and a circle of radius r with centre at the origin is given by the equation $x^2+y^2=r^2$. A curve having a node, as in Figure 1, could be described by the equation $y^2-x^2-x^3=0$. The one with the cusp, as in Figure 2, may be described by an equation such as $y^2-x^3=0$.

Just as a curve is an object in the plane and can be described by an equation $f(x,y)=0$ in two variables, a surface is an object in space and can be described by an equation $g(x,y,z)=0$ in three variables. For example $x^2+y^2+z^2=1$ and $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ are equations of a sphere and ellipsoid respectively (Figures 3 and 4). These surfaces have no singularity, because all points on them are 'like each other'.

The simplest surface with a singularity is the cone (Figure 5); its equation is $x^2-y^2-z^2=0$ and it has a singularity at the origin.

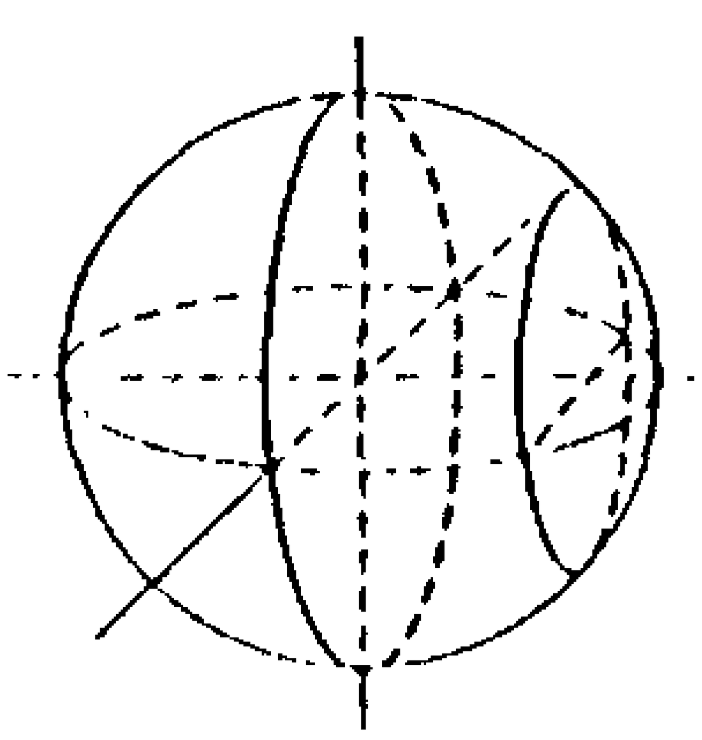

Figure 3.

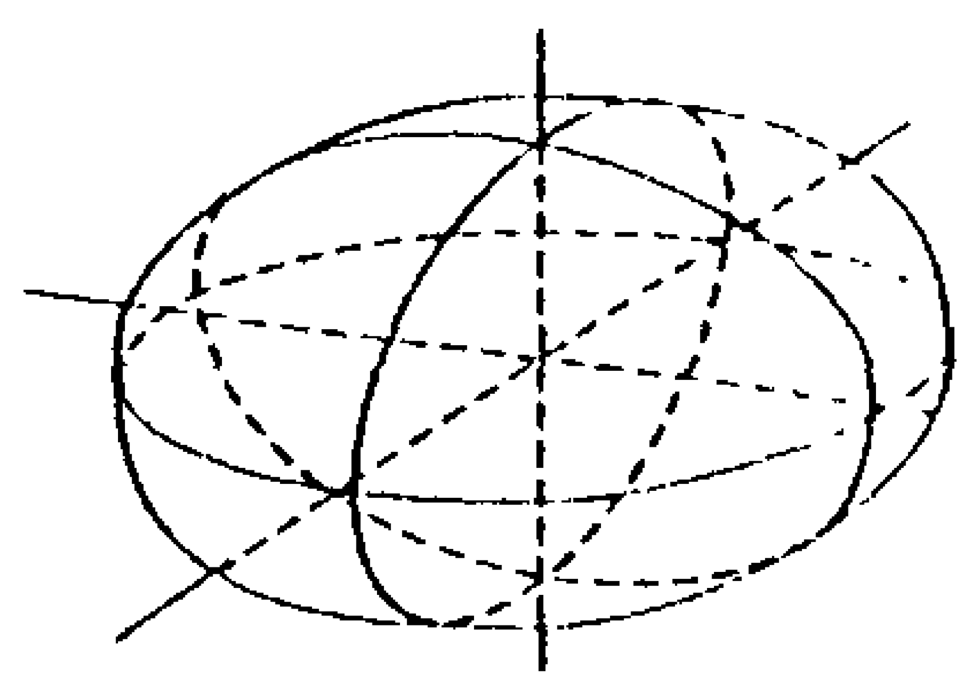

Figure 4.

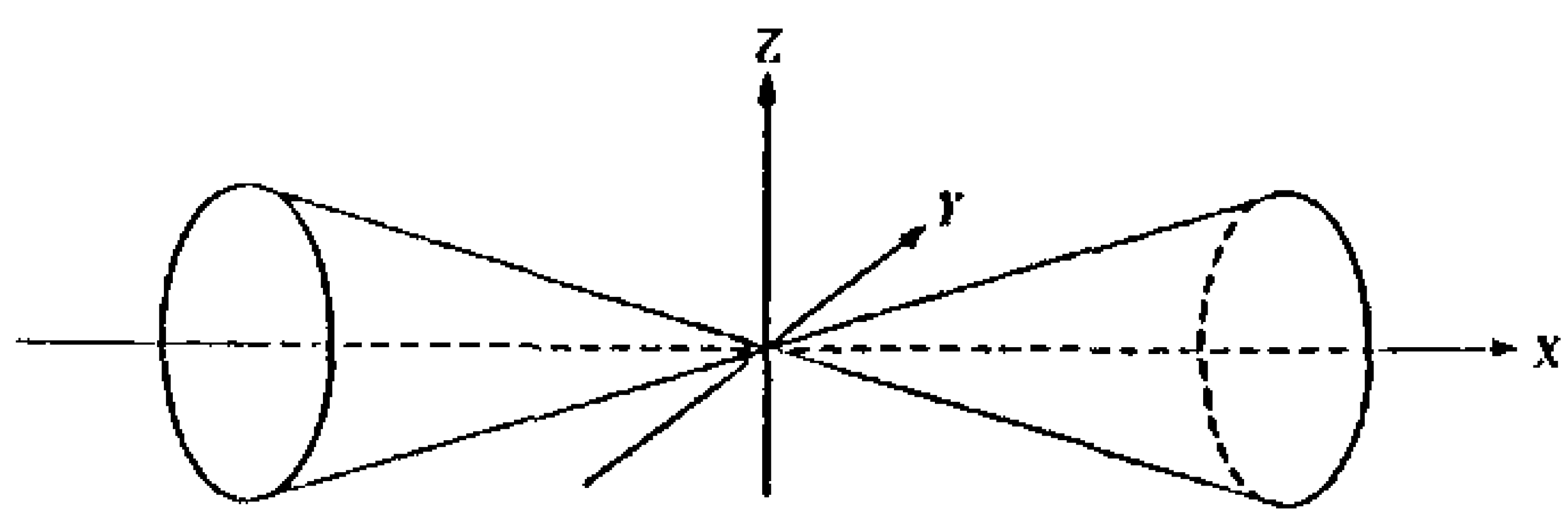

Figure 5.

2. What is resolution?

Resolving a singularity means making a transformation which will remove it. For example, let us make the quadratic transformation $x=x'$ and $y=x'y'$. Looking at the nodal cubic of Figure 6, we get $y^2-x^2-x^3=x'^2y'^2-x'^2-x'^3=x'^2(y'^2-1-x')$. Discarding the extraneous factor x'^2 we get the equation $y'^2-1-x'=0$ which is a parabola intersecting the y' axis in two points (Figure 7)

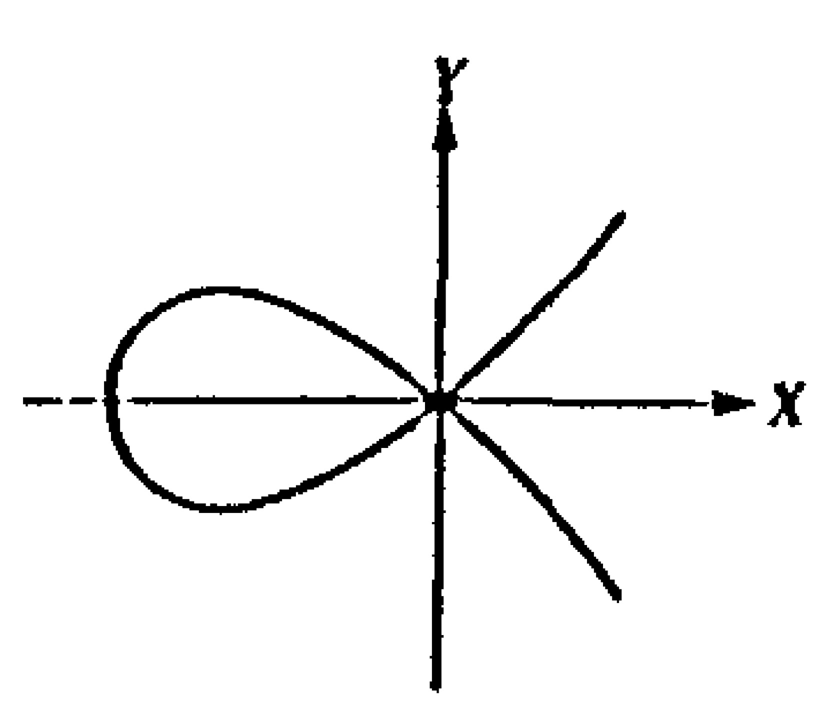

Figure 6.

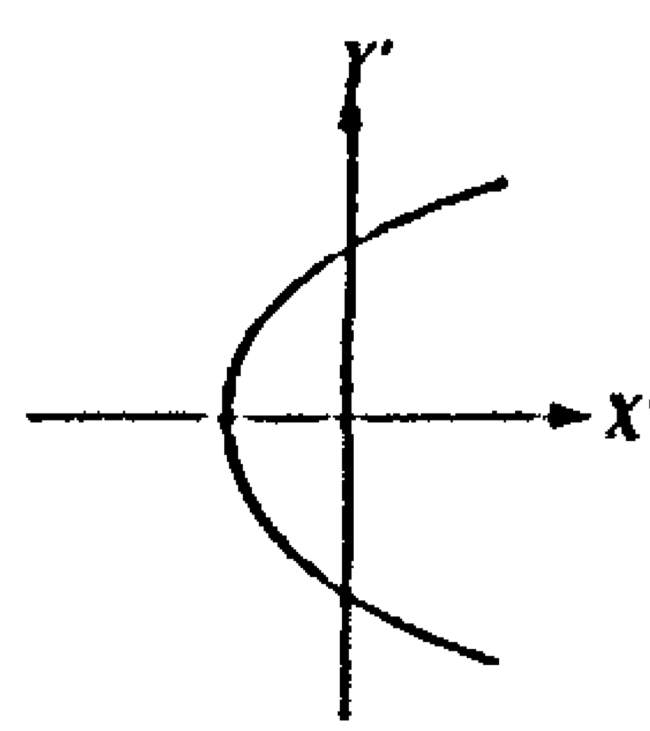

Figure 7.

which is nonsingular, i. e. has no singularities. Thus the quadratic transformation (QDT) has resolved the singularity of the nodal cubic.

Likewise, for the cuspidal cubic (Figure 8) we get $y^2 - x^3 = x'^2y'^2 - x'^3 = x'^2(y'^2 - x')$. Again, discarding the factor x'^2 we get $y'^2 - x' = 0$ (Figure 9) which is a parabola tangent to the y axis. So the QDT has resolved the singularity of the cuspidal cubic also.

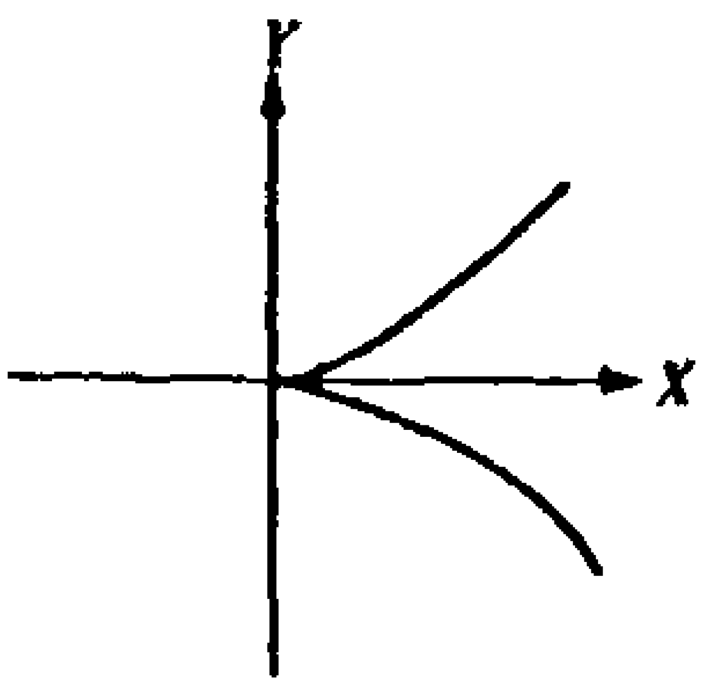

Figure 8.

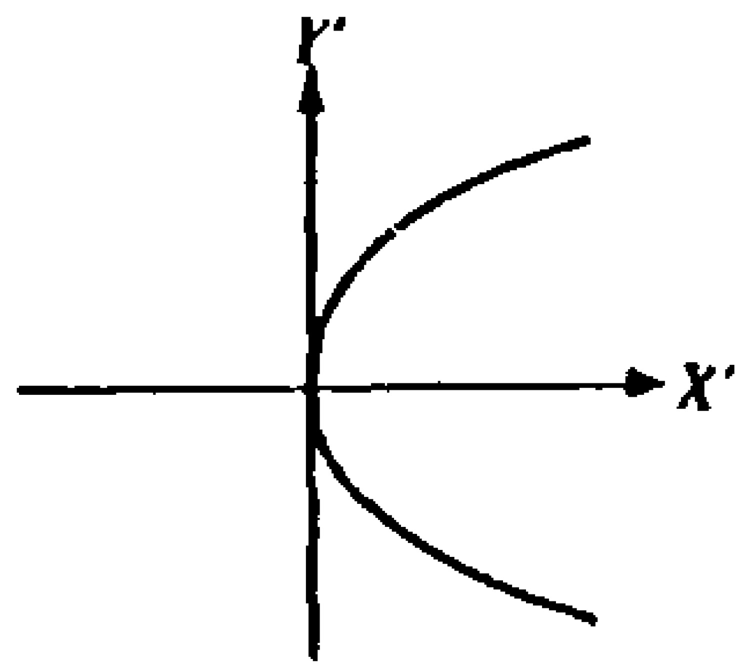

Figure 9.

Let us now consider the quintic (curve of degree five), $y^2 - x^5 = 0$ having a 'higher' cusp at the origin, see Figure 10. Making the QDT we get $y^2 - x^5 = x'^2y'^2 - x'^5 = x'^2(y'^2 - x'^3)$ and discarding the factor x'^2 we get the cuspidal cubic $y'^2 - x'^3 = 0$ (Figure 11). Thus the QDT did not resolve the higher cusp, but only transformed it into an ordinary cusp. One more QDT, $x' = x''$ and $y' = x''y''$, will of course resolve the ordinary cusp (Figure 12). Thus it takes two QDTs to get rid of the singularity of the cuspidal quintic. It was proved by Noether[1] around 1873 that the singularities of any algebraic plane curve $f(x, y) = 0$, where f is a polynomial of any degree can be resolved by a finite succession of QDTs. For a proof of this, and for an explanation why the factor x' may be disregarded, see Lecture 18 of my 1990 book[2].

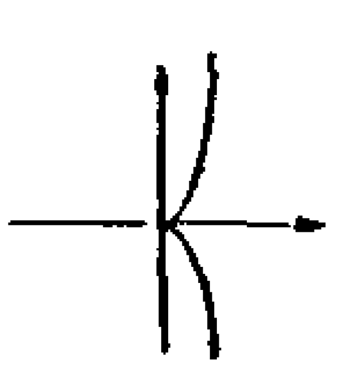

Figure 10.

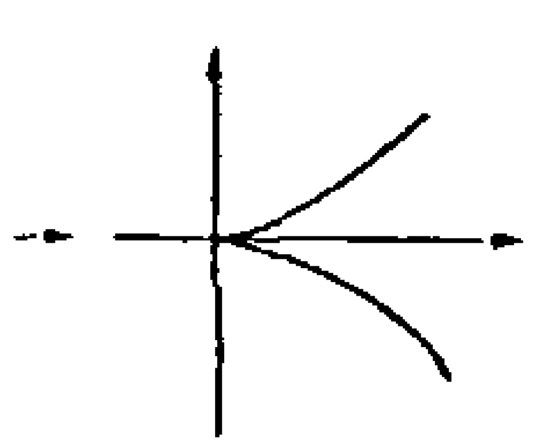

Figure 11.

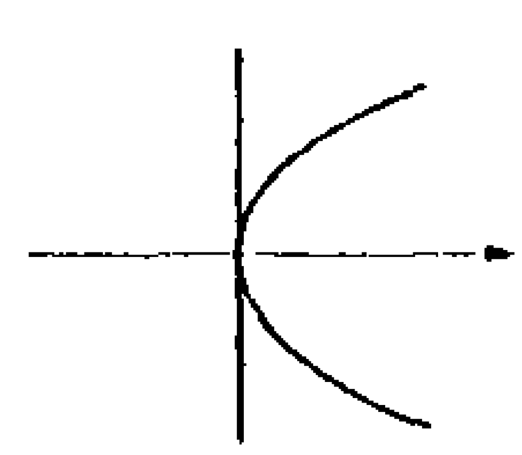

Figure 12.

Actually, the theorem of resolution of singularities of plane curves goes back to Riemann[3] who, around 1865, proved it by constructing the 'Riemann surface' of y as a function of x. Riemann's construction was highly analytic (that is based on complex analysis) and topological. Indeed, much of topology was invented by Riemann for this construction. Noether, who geometrized the resolution theorem of plane curves and who is sometimes called the father of algebraic geometry, was a disciple of Clebsch who himself was a follower of Riemann (see *Klein's History of Mathematics*[4] which is in German). Riemann himself was an early student of Gauss. It was Gauss' last pupil, Dedekind, who algebracized the resolution theorem for plane curves in his 1882 article in the *Crelle Journal* which he co-authored with Weber[5].

To deal with surface singularities, we can use QDTs of space such as $x = x', y = x'y', z = x'z'$. For example, by applying the above QDT to the cone, described in Figure 5, we get $x^2 - y^2 - z^2 = x'^2 - x'^2y'^2 - x'^2z'^2 = x'^2(1 - y'^2 - z'^2)$ and discarding the factor x'^2 we obtain the equation $1 - y'^2 - z'^2 = 0$ which represents a cylinder around the x'-axis. Clearly the cylinder has no singularities. Thus one QDT resolves the singularity of the cone. In general, in addition to QDTs, of space, we also require variations of them called MDTs of space, where we leave some of the variables alone, i.e. transformations such as $x = x', y = x'y', z = z'$. It was proved by Zariski[6] in 1943 that the singularities of any surface can be resolved by a finite succession of QDTs and MDTs.

Now QDTs, as applied to a surface (or curve), transform it to another surface (or curve), in an almost one-to-one manner. The same is true for MDTs applied to a surface. Quite generally, an almost one-to-one transformation is called a *birational transformation*. Algebraically speaking, a transformation is birational if its equations as well as the equations of its inverse are expressible in terms of rational functions, i.e. in terms of quotients of polynomial functions.

A correct proof of the theorem of resolution of singularities of surfaces, allowing general birational transformations, was first proved by Walker[7] in 1935. In turn Walker's proof was based on Jung's[8] 1908 paper on local uniformization, which is the local version of resolution of singularities.

3. Why resolution?

Now if a curve or a surface is nonsingular, i. e. if it has no singularities, then near every point it looks like the x-axis or the (x, y)-plane, and so, on the said curve or surface, we can carry out the operations of calculus such as differentiation and integration. This is one reason why we want to resolve singularities. Another reason is that the successive steps required to resolve a singularity do provide a lot of information about it, and putting together the resulting information for all the singularities tells us about various properties of the curve or the surface. An illustration of this can be found in Lecture 19 of my 1990 book[2], where it is shown how to calculate the 'genus' of a plane curve in terms of an analysis of its singularities, and in case the genus turns out to be zero, how to parametrize the curve by rational functions.

4. What is characteristic?

The above discussion of resolving singularities of curves and surfaces was originally restricted to characteristic 0, and its validity was extended to characteristic p by Hasse[9] and Schmidt[10] for curves around 1934, and by Abhyankar[11] for surfaces in his 1956 PhD thesis. So what is the difference between characteristic 0 and characteristic p? The difference lies in the coefficients of the equation $f(x,y)=0$ or the equation $g(x,y,z)=0$. If the coefficients are rational numbers, or real numbers or complex numbers then we are certainly in characteristic 0, because these 'fields', $1+1+1\cdots$ never equals 0 no matter how many times we add; these three number systems are fields because in them we can add, subtract, multiply and divide. On the other hand, for a prime number p, a field is of characteristic p if in it $1+1+\cdots+1(p \text{ times})=0$.

For example, the residue class ring $\mathbf{Z}/p\mathbf{Z}$ is a field of characteristic p. To explain this, let us recall that, briefly speaking, a *ring* is a set in which we can add, subtract and multiply; thus the set $\mathbf{Z}$ of all integers is a ring. A *field* is a ring in which we can divide by nonzero elements; thus rational numbers, real numbers and complex numbers are fields. The residue class ring $\mathbf{Z}/p\mathbf{Z}$ consists of boxes where in one box we put all integers which when divided by p leave the same remainder. The sum $A+B$ of two boxes A and B is the box containing $a+b$ with a in A and b in B; note that $A+B$ depends only on A and B and not on the elements a and b. The product of two boxes is defined in a similar manner. This makes $\mathbf{Z}/p\mathbf{Z}$ into a ring in which we can divide by nonzero elements, and so it is actually a field whose characteristic is obviously p.

5. Mixed characteristic or arithmetic case

Usually we write integers in their decimal expansion, i.e. as the sum of powers of ten with coefficients (called digits) ranging from zero to nine. For example, $423=(4\times 10^2)+(2\times 10)+3$.

Instead of ten we could use any integer $n>1$, and then we get n-adic expansion. This is especially significant when n is a prime number p. At any rate, the p-adic expansion of an integer is very similar to a polynomial in x; in this analogy, the prime number p plays the role of the variable x and the digits, which vary between zero and $p-1$, play the role of the coefficients. But because of carry-over, addition of p-adic expansions is not as straightforward as addition of polynomials. Likewise for multiplication.

Extending this further, consider a polynomial in a variable x, say $\varphi(x)=a_0+a_1x+\cdots+a_nx^n$, with integer coefficients $a_0,a_1,\ldots,a_n$. Replacing each coefficient a_i by its p-adic expansion, $\varphi(x)$ looks like a polynomial in x and p.

Similarly, a polynomial $f(x,y)$ in two variables x and y with integer coefficients may be construed to be a polynomial in three variables x,y,p with coefficients ranging between zero and $p-1$. Here p ranges over all prime numbers. In this manner, the original algebraic curve $f(x,y)=0$ becomes an *arithmetic surface* S. As we reduce the coefficients of $f(x,y)$ modulo p we get a curve C_p over the finite field $\mathbf{Z}/p\mathbf{Z}$ of characteristic p. Thus by imagining a fictitious variable z, we may say that the plane $z=p$ intersects the surface S in C_p. Thus S may be viewed as a 'family' of curves 'parametrized' by various primes. Since, for every prime number p, the surface S has points with coordinates in a field of characteristic p, the arithmetic case may be called the *mixed characteristic case*.

In a natural manner, these ideas lead to the problem of resolution of singularities of arithmetic surfaces. This problem was solved by Abhyankar[12] in 1965. Now in social life, uplift of a family is more than the well-being of all the individual members. Similarly, the resolution of singularities of an arithmetical surface, which is regarded as a family of curves of different characteristics, is something more than the simultaneous resolution of singularities of all the individual curves. In modern technical language, this amounts to resolving the singularities of a two-dimensional 'excellent scheme'; see Abhyankar's[13] 1968 lecture at the Tata Institute Colloquium.

Now just as a curve $f(x,y)=0$ over $\mathbf{Z}$ can be thought of as an arithmetic surface, a surface $g(x,y,z)=0$ over $\mathbf{Z}$ can be thought of as an *arithmetic solid*.

To resolve the singularities of an arithmetic solid is a nice challenge to an ambitious young student looking for a PhD thesis topic. The solution of this problem will not only be of obvious significance to algebraic geometry, but it will also be of considerable interest to number theory because it will amount to resolving the singularities of a family T of algebraic surfaces S_p, of characteristic p, parametrized by a varying prime number p.

6. Higher dimension

Having referred to an arithmetic solid, what is a usual (=algebraic=geometric) solid? Just as a surface in 3-space is given by an equation $g(x,y,z)=0$ in three variables, so a solid, or a three-dimensional *algebraic variety*, in 4-space is given by an equation $h(x,y,z,w)=0$ in four variables. Resolution of singularities of three-dimensional algebraic varieties, for characteristic zero, was achieved by Zariski[6] in 1944, and for characteristic p, Abhyankar[14] extended Zariski's proof in 1966.

More generally, for any positive integer n, an n-dimensional algebraic variety in $(n+1)$-dimensional space is given by an equation $\theta(x_1,x_2,\cdots,x_{n+1})=0$ in $n+1$ variables. For $n>3$, resolution of singularities of

n-dimensional algebraic varieties over a field of characteristic 0 was achieved by Hironaka[15] in 1964. For fields of characteristic p, this is a challenging open problem (D Sc *thesis*). Needless to say that, for *arithmetical varieties* of dimension > 3, it is even more of a challenge (F R S *thesis*).

7. Algebraic geometry

The various cases of resolution of singularities do occupy a central place in algebraic geometry. So what is algebraic geometry? Originally, algebraic geometry was simply a synthesis of the two subjects of analytic geometry and theory of equations. Analytic geometry is the study of geometric figures by algebraic equations, and was initiated by Descartes around 1635. Theory of equations, or high-school algebra, deals with simplifying expressions, factoring polynomials, making substitutions, and solving equations. This type of algebra started in India where it reached its climax in the hands of Bhaskaracharya in 1150. After that, via Arabia it went to Europe and got its biggest boost in the hands of Newton around 1660. Without getting deeper into this history, let me simply say that, in the last fifty years or so, algebraic geometry seems to have taken off into rarefied abstractions, leaving behind its concrete manipulative origins.

But personal experience has convinced me that reverting to the classical high-school algebraic origins of algebraic geometry is very desirable for tackling problems like the problem of resolution of singularities.

8. Scientists and engineers

So I am very happy to see that, during the last dozen years or so, a return to such concrete origins of algebraic geometry has been inspired because of the interest taken by scientists and engineers. This motivated me to give a series of lectures to an engineering audience and compile them into a book on *Algebraic Geometry for Scientists and Engineers*[2]. Needless to say that for further information concerning the matter dealt with in this paper, reading of this book would be quite profitable.

In the said book, in addition to discussing the method of concrete or high-school algebra, I also relate it to the language of abstract or college algebra, because it too can be quite useful. An illustration of college algebra is Dedekind's proof of resolution of singularities of plane curves referred to in the second section. What Dedekind does is to take the ring R of polynomial functions on the curve and then pass to its integral closure S in the field of rational functions on the curve. Now the curve whose ring of polynomial functions is S is automatically nonsingular and so we are done!

Having talked about high-school algebra and college algebra, let me close by saying that the third level of algebra, i.e. the super-abstract or university algebra, is also sometimes useful. The notion of excellent schemes cited in the fourth section is a sample from university algebra.

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