Affine geometry

N. Mohan Kumar

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

1. Some definitions

Affine geometry is the study of affine varieties. An affine variety is the set of common zeroes of a collection of *polynomial functions* in $\mathbf{C}^n$, the *affine n-space* over the complex numbers (or more generally k^n, k any algebraically closed field). For example the set of points of the form $(t^2, t^3) \in \mathbf{C}^2$, where $t \in \mathbf{C}$, is an affine variety, since it is the zeroes of the single polynomial $f(x, y) = x^3 - y^2$, where x and y are the co-ordinate functions on $\mathbf{C}^2$. If $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^m$ are two affine varieties, then a map of affine varieties $f: X \to Y$ is called a *morphism* if it is given by a polynomial map $F: \mathbf{C}^n \to \mathbf{C}^m$ such that $F(X) \subset Y$. Any map $F: \mathbf{C}^n \to \mathbf{C}^m$ is given by an m-tuple $(F_1, \ldots, F_m)$ where F_i's are functions on $\mathbf{C}^n$ (i.e. are maps $\mathbf{C}^n \to \mathbf{C}$). So once we fix the co-ordinates, F_i's are functions in n variables. What we insist on is that, we consider only F_i's which are polynomials in these variables.

Example 1.1: Consider $\mathbf{C} \xrightarrow{f} \mathbf{C}^2$, where $f(t) = (t^2, t^3)$. This gives a morphism $\mathbf{C} \to W = \{X^3 - Y^2 = 0\}$.

A morphism $f: X \to Y$ is an isomorphism if there exists a morphism $g: Y \to X$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$. In the above example one can see that f is a bijection of sets but f is *not* an isomorphism. So isomorphisms are more than bijective morphisms. (Topologically we cannot distinguish $\mathbf{C}$ from W, but as algebraic varieties they are distinct.)

Instead of W, if we considered a small deformation, $W_\varepsilon: \{X^3 - Y^2 = \varepsilon\} \subset \mathbf{C}^2$, then one can see that there is no non-constant morphism $f: \mathbf{C} \to W_\varepsilon$. Though one may verify this by hand in this simple case, less trivial examples would need the powerful machinery of projective varieties to analyse. (In this particular case it will be easier if we knew what the *genus* of a curve is.) We will not go into it here, except to warn the reader that often deep results in affine geometry will use results from projective geometry and both these subjects should be considered part of a bigger whole—algebraic geometry.

With the above easy definitions, we already come across a natural problem, which even in fairly simple cases, is not well understood at present. Let $X \subset \mathbf{C}^n$ and $Y \subset \mathbf{C}^n$ be two affine varieties. Assume they are isomorphic. Then by definition, we have maps F, G: $\mathbf{C}^n \to \mathbf{C}^n$ such that $F(X) \subset Y$ and $G(Y) \subset X$ and the induced maps $f = F_{|X}$ and $g = G_{|Y}$ are isomorphisms. We may ask whether we can choose F and G to also be isomorphisms? (Of course if we can choose F to be an isomorphism, then we can choose $G = F^{-1}$.) If X is a point then since any point can be taken to any other point by an automorphism of $\mathbf{C}^n$ we are through. So let us take the next simplest case: let $X \subset \mathbf{C}^2$ be an affine variety isomorphic to the complex line $\mathbf{C} \subset \mathbf{C}^2$. Does there exist an automorphism of $\mathbf{C}^2$ which takes X to $\mathbf{C}$? Already such an innocuous-looking problem is very hard.

Example 1.2: Let X be defined by the equation

$$x + y + x^2 + y^2 + 2xy^2 + y^4 = 0.$$

Then one can show that $X \cong \mathbf{C}$. Does there exist an automorphism of $\mathbf{C}^2$ which takes X to the line defined by $x = 0$? One can do this by hand easily if you know the right principle.

This principle enabled Abhyankar and Moh[1] (for a self contained account see ref. 2) to solve the above problem in the affirmative as recently as two decades ago. The proof, needless to say, is very intricate. There are many other results known in this direction. But, for example, even the case of $\mathbf{C}$ in $\mathbf{C}^3$ is as yet intractable.

Before we get carried away, I want to stress that the problem of curves in 3-space is more meaningful in this generality than say the case of curves in 2-space. For instance, consider the two curves $xy - 1 = 0$ and $x^2y^3 - 1 = 0$ in $\mathbf{C}^2$. One can easily see that both these curves are isomorphic. Take for instance, $F(x, y) = (x^3, y^2)$ and $G(x, y) = (x^2, y^3)$. But there does not exist an isomorphism $A: \mathbf{C}^2 \to \mathbf{C}^2$ which takes one curve into another. This can be seen by the following two fairly obvious facts:

1. If $F = (f, g)$ is any automorphism, then the Jacobian matrix

$$J(f, g) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

has determinant which is a polynomial with no zeroes and hence it is a non-zero constant.

2. Since F is an automorphism, f, g can be chosen as coordinates of $\mathbf{C}^2$ as well and in particular neither of them can be a product of two non-constant polynomials (i.e. f, g are *irreducible* polynomials).

But it was proved by Cowsik and Nori[3] (see, e.g. ref. 4) that if C is any smooth (read *manifold*) curve embedded in two different ways in $\mathbf{C}^n, n \geq 4$, then there exists an automorphism of $\mathbf{C}^n$ taking one to the other. (The result is much more general, but we will not go into it here.)

The previous discussion throws up yet another natural and tantalizing problem the answer to which is not known. We saw that if $F: \mathbf{C}^2 \to \mathbf{C}^2$ is an automorphism, then the Jacobian deteminant is a non-zero constant. What about the converse:

If $F=(f,g): \mathbf{C}^2 \to \mathbf{C}^2$ (or more generally $\mathbf{C}^n \to \mathbf{C}^n$) is a morphism whose Jacobian determinant is a non-zero constant, is F an automorphism?

This is the famous 'Jacobian conjecture'. This problem is notorious in the sense that many well-known mathematicians have worked on it in the last several decades and several alleged proofs have been published. All of them turned out to be wrong. The inverse function theorem of calculus will tell us that the condition on the Jacobian matrix is what we need to get a *local inverse*. The problem is to find a *global inverse*. The answer to the question is in the negative if we allow ourselves entire functions, not just polynomials. For example we can take $F=(e^x, e^{-x}y)$. Then the Jacobian determinant is 1, but the map is not even injective. The conjecture has been verified for polynomials f, g up to very large degrees. One reason for the difficulty is that the structure of automorphisms of $\mathbf{C}^n$ can be fairly complicated. Automorphisms of the complex line $\mathbf{C}$ are fairly easy to understand. They consist of linear maps $z \mapsto az+b, a, b \in \mathbf{C}, a \neq 0$. Already for $\mathbf{C}^2$, we have nonlinear automorphisms of the following kind:

$$x \mapsto ax+b \quad \text{and} \quad y \mapsto cy+f(x) \quad a, c \neq 0,$$

where f is any polynomial in x. It is known classically that all automorphisms of $\mathbf{C}^2$ are composites of such automorphisms. Once we go to higher dimensions even this is not known. These are very fascinating and important problems in affine geometry. There are specific examples of automorphisms of $\mathbf{C}^3$, constructed by Nagata[5], which we do not know to be *elementary* in the sense that they are composites of automorphisms of the above type.

There is another notion of 'elementary' in algebra which often comes up in many situations. Let A be any commutative ring (with 1). For example,

1. $A=\mathbf{C}[x,y]$, the ring of polynomials,
2. $A=\{a+ib|a,b$ are integers$\}$, the ring of Gaussian integers.

Let $SL(2,A)$ be the group of 2×2-matrices over A with determinant 1. Consider matrices of the form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

and take the subgroup generated by such matrices in $SL(2,A)$. These matrices are called elementary matrices and the subgroup generated by them is denoted by $E_2(A)$. (A similar notion can clearly be defined for any $SL(n,A)$.)

Example 1.3:

1. If $A=$ a field, say $\mathbf{C}$, then $SL(n,A)=E_n(A)$.
2. Let $A=\mathbf{C}[x,y]$, polynomial ring in two variables. Let

$$M=\begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}.$$

Then $M \in SL(2,A)$ and one can show that M is not elementary.
(However there is a deep theorem of Suslin[6] which says that, if $A=k[x_1,\ldots,x_m]$, k a field, then $SL(n,A)=E_n(A)$ if $n \geq 3$.)

Now the structure of automorphisms of $\mathbf{C}^2$ implies that if $F=(f,g)$ is an automorphism with det $J(F)=1$, then $J(F)$ is actually elementary in $SL(2,\mathbf{C}[x,y])$. This was shown by Wright[7]. He also showed that if $F: \mathbf{C}^2 \to \mathbf{C}^2$ is any morphism with $J(F)$ elementary, then F is an automorphism. Unfortunately these and other advances in the theory have not yet led to a solution of the Jacobian conjecture.

2. Equations defining varieties

Now I want to look at a set of problems with a different flavour. Granting that our aim is to study affine varieties, it is clearly important to study the equations defining them. If $X \subset \mathbf{C}^n$ is defined as the zero set of say $f_1,\ldots,f_k$, then for any polynomials g_i, the polynomial $F=\Sigma g_i f_i$ also vanishes on X; hence X is also defined as the zero set of $f_1,\ldots,f_k$, F. Clearly adding on this F has no effect. But it illustrates two things:

1. The equations defining X may not be unique. There could be an entirely different set of polynomials (a 'better' set, perhaps) defining the same variety.
2. The set of all functions vanishing on X is closed under the operations of addition and multiplication by any function. In algebra this is the same as saying that

$$I(X)=\{\text{the set of all functions vanishing on } X\},$$

is an *ideal* in $R=\mathbf{C}[x_1,\ldots,x_n]$.

Given these facts, the classical *Hilbert basis theorem* (see ref. 8 for example) will tell us that there exists a finite set $f_1,\ldots,f_k \in I(X)$ such that any element $F \in I(X)$ can be written as a linear combination $F=\Sigma g_i f_i$ with $g_i \in R$. In particular, any affine variety is defined by a finite set of polynomials. Now it is natural to ask the following:

What is the smallest number of equations required to define a given variety in $\mathbf{C}^n$?

There are some subtleties which make this question not well posed. For example, if we take the complex line $\mathbf{C}$, with co-ordinate function x, then $x=0$ and $x^2=0$ both define the origin. Do we consider these two varieties to be the same or distinct? There are reasons to keep track of this although in this simple example this is not evident. So, let us work out a more complicated example.

Example 2.1: Consider the curve $X=\{(t^3,t^4,t^5)\in\mathbf{C}^3\}$, $t\in\mathbf{C}$. Let x,y,z be the co-ordinates of $\mathbf{C}^3$. Let us consider the 2×3 matrix,

$$M=\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}$$

(I wrote this down because I have information which you may not have.) Then any 2×2 minor of M vanishes on X. Further any polynomial which vanishes on X will be a linear combination in $R=\mathbf{C}[x,y,z]$ of these minors, i.e. $I(X)$ is generated by these three elements. It can also be shown that $I(X)$ cannot be generated by a fewer number of elements. But if we look at the common zeroes of the two polynomials, $F=xz-y^2$ and

$$G=\det\begin{pmatrix} x & y & z \\ y & z & x^2 \\ z & x^2 & 0 \end{pmatrix}$$

then one can verify that it is precisely X. Of course F,G do not generate $I(X)$, but they define X at least set-theoretically. So in our question, we had better specify whether we are looking for the number of generators of $I(X)$ or just the number of equations required to define X set-theoretically, since these numbers can be different.

So we will distinguish between these numbers by adding the prefix *ideal-theoretic* (or *scheme-theoretic*) and *set-theoretic* and use the notation $\mu(X)$ and $\mu_s(X)$ respectively. It is clear that $\mu(X)\geq\mu_s(X)$. In the above example, $\mu(X)=3$ and $\mu_s(X)=2$.

Before we set about trying to compute $\mu(X)$ we should know what properties of X will influence $\mu(X)$, what data will give us some clue. The crudest datum we can start with is the *dimension* of X. Intuitively everyone knows what dimension means and we shall not try to make it precise here. For example affine n-space has dimension n. What we have been calling curves have dimension 1. Loosely speaking, we may do the following:

Let $X\subset\mathbf{C}^n$. If they are equal, then the dimension is of course n. If not, we may change variables so that the point $P=(0,\ldots,1)\notin X$ and $Q=(0,\ldots,a)\in X$ for some a. Now look at Y= image of X in $\mathbf{C}^{n-1}$ under the projection

$$(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_{n-1}).$$

Then the closure of Y is an affine variety and we call $\dim X$ = dimension of this closure. Continuing like this we can define the dimension of any affine variety. Of course, a lot of things have to be checked to make this a good definition, in particular that this number does not depend on the projection that we have chosen. We will skip over this. Yet another intuitive way is to look at a point $P\in X$, where X is a manifold. That such points exist can be proved. Having found one such, we may find local co-ordinates $x_1,\ldots,x_n$ of $P\in\mathbf{C}^n$, such that X is locally defined by the vanishing of $x_1=x_2=\cdots x_k=0$. Then define $\dim X=n-k$.

If X is defined by r equations in $\mathbf{C}^n$, then since each equation can reduce the dimension at most by one, $\dim X\geq n-r$. Thus we get a crude estimate $\mu_s(X)\geq n-\dim X$. We say that X is a (ideal-theoretic) *complete intersection* if $\mu(X)=n-\dim X$ and a *set-theoretic complete intersection* if $\mu_s(X)=n-\dim X$. So the curve $\{(t^3,t^4,t^5)\in\mathbf{C}^3\,|\,t\in\mathbf{C}\}$ is a set-theoretic complete intersection but *not* an ideal-theoretic complete intersection. A question attributed to Kronecker is the following:

Is every curve $X\subset\mathbf{C}^3$ a set-theoretic complete intersection?

Complete intersection varieties have many desirable properties. But to recognize one as such is not easy. Kronecker's problem is completely solved in positive characteristics (whatever they are) (ref. 3) but has withstood all onslaughts in its original form over $\mathbf{C}$. This is not to say that all attempts have been futile. At least when the curve in question is a manifold, it is known, due to Ferrand and Szpiro[9] that it is a set-theoretic complete intersection. Their proof goes along the lines of the proof we described in the earlier example.

If $X\subset\mathbf{C}^3$ is a smooth curve (i.e. a manifold), then one shows that it is ideal-theoretically defined by the 2×2 minors of a 2×3 matrix. (That it is defined by the $r\times r$ minors of some $r\times(r+1)$ matrix is easier and is valid for any curve, not just smooth curves.) Let us call this matrix $M=(a_{ij})$. The main point is that in a suitable sense we can assume the first 2×2 matrix to be *symmetric*, i.e. $a_{21}=a_{12}$. (This is not strictly true.) But if we assume this, then consider the 3×3 matrix,

$$N=\begin{pmatrix} & M & \\ a_{13} & a_{23} & 0 \end{pmatrix}.$$

Then the two equations $a_{11}a_{22}-a_{12}^2$ and $\det N$ will define X set-theoretically exactly as in the example. The Ferrand–Szpiro proof is to make this argument rigorous. Unfortunately it requires techniques beyond the scope of this article.

At this point it is legitimate to ask the question, whether there is any variety $X \subset \mathbf{C}^n$, which is smooth, but not a complete intersection? We will touch upon this in the next section.

3. Certain intrinsic invariants

To systematically study the problem stated in the previous section, it becomes necessary to understand properties of complete intersection varieties. These intrinsic properties might suggest methods to recognize when a variety is a complete intersection. For example we have already seen an intrinsic invariant called the dimension. For instance, if we took say, the union of a curve and a surface, our definition of complete intersection itself is unclear. We also saw that the curve

$$X=\{(t^3,t^4,t^5)\in \mathbf{C}^3\,|\,t\in \mathbf{C}\}$$

is not a complete intersection. One can see in this example that in no neighbourhood of the origin can we define X by two equations. This is a *local invariant.* This difficulty disappears if we assume that our variety is smooth. Even after assuming that the variety is smooth, there might be invariants which have special values for complete intersections. Our aim is to look for such invariants or *obstructions* which when $\neq 0$ show that the variety is not a complete intersection; then you might hope that if these are indeed zero, then the variety might be a complete intersection. Of course, these must be *coarse* enough to be of some use and not just tautologies, but *fine* enough to distinguish complete intersections among affine varieties. Finally it is only such systematic studies which will reveal the mysteries.

Let us again go back to a smooth curve X in $\mathbf{C}^3$. X is a one-dimensional complex manifold (i.e. *Riemann surfaces*). So locally, $X \cong U \subset \mathbf{C}$, U open. So all 1-forms on X are of the form $f dz$, f a locally holomorphic function and z a local co-ordinate. Thus the *bundle* of 1-forms, denoted by Ω^1_X and called the cotangent bundle, is locally generated by one element. Now, assume that X is a complete intersection of two equations, say f and g. Consider the Jacobian matrix,

$$J=\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix},$$

where x, y, z are the co-ordinates of $\mathbf{C}^3$. Since X is a manifold, the implicit function theorem implies that at any point $P\in X$, at least one of the 2×2 minors of J does not vanish. Since this is true at all points and the minors are *global functions*, we may appeal to a well-known theorem *Hilbert's nullstellensatz* (which is intuitively not hard) (see for example ref. 8) to see that there exist global functions a, b, c such that the function $A=aJ_{12}+bJ_{23}+cJ_{31}$, where J_{ij} are the minors of J, does not vanish on X. Then one sees that the 1-form $\omega=a\,dx+b\,dy+c\,dz$ does not vanish anywhere on the curve. So if X is a complete intersection, then it must have a nowhere vanishing 1-form (globally). This is a stringent restriction. Now we know what we are looking for, it is not very hard to construct examples of $X\subset \mathbf{C}^3$ which do not possess such 1-forms and thus we get examples of non-complete intersection curves. The existence of a nowhere vanishing 1-form fortunately does imply that X is complete intersection.

This aside, my point here is that an intrinsic invariant has at least told us what a necessary condition is for a curve to be complete intersection. These type of invariants are called *Chern classes*. For a smooth curve X, the property that X has a nowhere vanishing 1-form (or equivalently Ω^1_X is singly generated) is generally termed as its 1st Chern class being zero, written $c_1(\Omega^1_X)=0$.

Let me explain this a little more by the following related problem. We know that any point $P\in \mathbf{C}^n$ is a complete intersection. (If $P=(\alpha_1,\dots,\alpha_n)$ then $I(P)=\{(x_1-\alpha_1,\ x_2-\alpha_2,\dots,x_n-\alpha_n)\}$.) Instead assume that $P\in X\subset \mathbf{C}^n$, where X is a smooth affine variety of dimension d. One might ask whether there exists d functions $f_1,\dots,f_d$ on X such that their common zero set is precisely P. (We then say that P is a complete intersection in X). To be specific, let us consider $X=\{y^2-x^3-1=0\}$ in $\mathbf{C}^2$ and $Y=\{x^3+y^3+z^3-1=0\}$ in $\mathbf{C}^3$. X is a smooth curve and Y is a smooth surface. It so happens that a general point $P\in X$ is *not* a complete intersection on X but it is so on Y. There seems to be no clue as to what is going on. Of course, if one tries to do this by brute force one may succeed. But it will not satisfy the aesthetic sense of the mathematician, who will look for a more elegant approach, albeit not so elementary. One approach is as follows. On any curve X let us first take the abelian group generated by formally adding points of X. Let us call this $F(X)$. So a typical element of $F(X)$ is of the form $\Sigma a_i P_i$, $a_i\in \mathbf{Z}$, $P_i\in X$, points. The sum is just a formal sum. If f is any non-zero function on X and $P\in X$ is such that $f(P)=0$, one may define $v_P(f)\in \mathbf{Z}$ to be the order of vanishing of f at P. Now given f, one has an element $\Sigma v_P(f)\cdot P\in F(X)$. Notice that, this sum is finite, since f can vanish at only finitely many points. These elements form a subgroup of $F(X)$. Let $CH_0(X)$ be the quotient of $F(X)$ modulo this subgroup, the so-called *Chow group of zero cycles* of the curve X. If $P\in X$ then it defines an element of $CH_0(X)$. This is the Chern class of P. Further if f generates $I(P)$, we see that the class of P is zero in $CH_0(X)$. Conversely if the class of a point P is zero in $CH_0(X)$, then the point is defined by the vanishing of a single function. So to prove that a general point $P \in X$ is not a complete intersection, one needs only to show that $CH_0(X) \neq 0$. This in general requires techniques from projective geometry. The fact is that the curve X is an *elliptic curve* and they have very large Chow groups.

A similar definition can be made for any variety of any dimension. If X is an arbitrary variety, define $CH_0(X)$ to be the quotient of $F(X)$ by the subgroup generated by elements $\Sigma_{P \in Y} v_P(f)$, where f runs over all non-zero functions on Y and Y runs over all curves on X. It will follow by definition, that if a point is a complete intersection then its Chern class is zero. The converse is true for affine varieties over **C**, a theorem due to Murthy[10]. The proof requires deep analysis of what are known as *vector bundles* (or *projective modules*) over these varieties. The point I want to stress is that it is such unifying general theories which ultimately pay rich dividends.

Example 3.1: We will conclude this article by showing that the Chow group of Y, the cubic surface we defined earlier is zero.

First let us prove that the Chow group of a plane curve of degree atmost two is zero. So let $X = \{f(x,y) = 0\}$ be a plane curve and assume that $\deg f \leq 2$. If $\deg f = 1$, then $f = 0$ is a line and hence any point on X is defined by the vanishing of one function. Thus by definition the Chow group is zero. So let us assume that $\deg f = 2$. If f is a product of two non-constant polynomials, then both must be of degree 1 so that X is a union of two lines and since any point on X must lie on one of these lines we are done. So let us assume that f is irreducible. Then by a change of variables, we can assume that $f = x^2 - g(y)$, where $\deg g \leq 2$. Now again by a change of variables, we can assume that $f = x^2 - y$ if $\deg g = 1$ and if $\deg g = 2$ then after a further change of variables we can assume that $f = xy - 1$. In either case if $P = (a,b) \in X$, then it is the set of zeroes of the single polynomial $x - a$. So the Chow group of X is zero.

Now let us go back to the cubic surface $Y = \{f = x^3 + y^3 + z^3 - 1 = 0\}$. Let $P = (a,b,c)$ be any point on Y. We will only treat the case when $a + c \neq 0$ and $b \neq 1$, the rest of the cases being similar. Let d be chosen so that $a + d(b-1) + c = 0$. Consider the intersection, denoted by X, of the plane $x + d(y-1) + z = 0$ with Y. By choice, $P \in X$. Substituting the expression for z from the linear equation, we get the equation g defining X in the (x,y)-plane. Since f is cubic so is g. It is clear that $g(x,y) = (y-1)h(x,y)$, where h is of degree 2. Since $b \neq 1$, $P \in C = \{h = 0\} \subset Y$. But the Chow group of C is zero by the previous paragraph and hence the class of P in the Chow group of Y is zero. Since P was an arbitrary point we get that $CH_0(Y) = 0$.

For a more detailed account with proofs of many of the above discussions, the reader may see ref. 11.

1. Abhyankar, S. S. and Moh, T. T., *J. reine. angew. Math.*, 1975, **276**, 148–166.
2. Abhyankar, S. S., TIFR Lecture notes, Bombay, 1977, (Notes by B. Singh)
3. Cowsik, R. C. and Nori, M. V., *Invent. Math.*, 1978, **45**, 111–114.
4. Srinivas, V., *Math. Ann.*, 1991, **289**, 125–132.
5. Nagata, M., AMS Regional Conf. Ser. 1978, 37.
6. Suslin, A. A., *Math. USSR—Izvestija*, 1977, **11**, 221–238 (English translation).
7. Wright, D., *J. Pure Appl. Algebra*, 1978, **12**, 235–251.
8. Lang, S., *Algebra*, Addison–Wesley, 1971.
9. Ferrand, D., *C. R. Acad. Sci. Paris*, 1975, **281**, 345–347.
10. Murthy, M. P., *Bull. Am. Math. Soc.*, 1988, **19**, 315–317.
11. Ohm, J., *M. A. A. Studies in Mathematics* (ed. Seidenberg, A.), vol. 20, pp. 47–115.

Projective algebraic varieties

V. Srinivas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

1. Projective geometry

In their famous book *Geometry and the Imagination*, Hilbert and Cohn-Vossen[1] describe projective geometry as a study of 'geometrical facts that can be formulated and proved without any measurement or comparison of distances or angles'. They give the following example:

> ... if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure.

To give a physical analogy, imagine the point of projection as a light source, and the first planar figure