In this issue

Algebraic geometry

Algebraic Geometry forms a special section in this issue. This was prompted by the fact that two of our mathematicians, M. S. Narasimhan and C. S. Seshadri, who have contributed much to Algebraic Geometry, are sixty this year. We are thankful to V. Srinivas of TIFR, the Guest Editor, for the effort he took in this regard.

Starting with the conic sections studied by the Greeks, algebraic geometry arises out of coordinate geometry, through the study of the solutions of a system of polynomial equations in several variables. It is now a vast subject with inputs from (and applications to) complex analysis, commutative algebra, topology and number theory.

The following series of articles gives an introduction to some parts of modern algebraic geometry, and in particular, to topics where Indian mathematicians have made some contribution.

Affine algebraic varieties are the building blocks of algebraic geometry, and Mohan Kumar's article (**page 218**) gives an introduction to their study. Problems about affine varieties can often be stated in very concrete terms, as he illustrates. In many ways, however, it was found that affine varieties need to be 'completed' by adding 'points at infinity', in order to obtain a more coherent and aesthetically satisfying theory; this leads to the notions of projective space and projective varieties, which are the main objects of study in algebraic geometry. These are introduced in Srinivas' article (**page 222**).

Among algebraic varieties, the ones with the most pleasant properties are non-singular projective varieties, since one is able to do integral and differential calculus on them; this leads to a deeper understanding of these. A general variety may have singularities; the problem of resolution of singularities is the problem of 'modifying' a singular variety in simple ways repeatedly so as to obtain a non-singular one. If this can be done, then in principle, most of the properties of the singular variety can be reduced to properties of smooth varieties, which are 'simpler'. This important problem is discussed in Abhyankar's article (**page 229**).

Paranjape's article (**page 233**) explores the connections between algebraic curves, completely integrable dynamical systems and the theory of soliton equations (like the famous KdV equation). This became an active area in algebraic geometry in the seventies, and led (among other things) to the solution of the famous Schottky problem. These ideas are currently of interest in string theory.

The Weil conjectures, discussed in Srinivas' article (**page 239**), give a remarkable relationship between topology and number theory. These conjectures were proved in the sixties and seventies, and are one of the remarkable successes of 'abstract' algebraic geometry as developed by Grothendieck and his followers. This work was a forerunner to the solution by Faltings in 1983 of the famous Mordell conjecture in the theory of Diophantine equations. One of the exciting areas of research today is arithmetic algebraic geometry, where one seeks deeper connections between number theory on the one hand, and topology and complex analysis on the other. The Weil conjectures represent the first step in this direction.

The cover picture

The special section being on algebraic geometry we thought it would be in the fitness of things to have a picture of Diophantus on the cover. All mathematicians starting from Viète (1846) pay homage to this ancient Greek mathematician as the one who really laid the foundations of this fascinating subject. We were completely unsuccessful in getting a reproduction of a drawing, painting, sculpture of this remarkable Greek. But the only thing connected with Diophantus that survives today is his famous treatise *Arithmetica*. Even here only six of the thirteen chapters announced by him in the introduction survive. With wishful thinking the historian of mathematics André Weil remarks: "Important cuneiform texts *may* still be buried in Mesopotamia and even more probably (according to Neugebauer) in the dusty basements of our museum!"

We succeeded in getting a picture of the title page of *Arithmetica* translated into Latin, published in 1670, which says: 'Six books of *Arithmetica* and one book on polygonal numbers by Diophantus of Alexandria.'

Pierre de Fermat (a judge of Toulouse)—to whom also is attributed the starting of different disciplines, e.g. (a) analytical geometry (with Descartes), (b) inifinitesimal calculus (with Leibnitz and Newton), and (c) probability theory (with Pascal)—read a translation of *Arithmetica*. In the margin next to the Pythagorian problem of finding squares that are sums of two other squares, he wrote in Latin:

> On the other hand it is impossible for a cube to be a sum of two cubes, a fourth power to be a sum of two fourth powers or, in general, for any number that is a power greater than second to be the sum of two like powers. *I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain.*

Most mathematicians of today do not believe that Fermat had a correct proof! But Fermat's *conjecture* is found to be empirically true for all the numbers tried so far (up to a million digits), but *Fermat's last theorem* is yet to be proved.

André Weil's autobiography

Richard Askey (of Wisconsin), one of the triumvirate associated with the so-called *Ramanujan's Last Notebook*, informed us that André Weil's autobiography in English had appeared in the United States and suggested that *Current Science* should review it. We were able to persuade Raghavan Narasimhan (of Chicago), one of our distinguished mathematicians, to write a review of the book *André Weil: The Apprenticeship of a Mathematician*", which we reproduce on **page 247**. André Weil, Professor-Emeritus at the Institute of Advanced Studies, Princeton, is one of the leading mathematicians of this century, known for his work in number theory and algebraic geometry. We thought it would be appropriate to have a review of his book in this theme issue on algebraic geometry. In 1937 E. T. Bell wrote of Gauss: 'He lives everywhere in mathematics.' André Weil, it is said, was inspired by some of the work of Gauss on the number of solutions to polynomial equations modulo a prime number and in 1947 made three far-reaching guesses known as 'Weil's conjectures' (**page 239**). In these he gave formulae for the number of solutions to an algebraic equation in a finite field. These allow one to deduce whether a given equation does or does not have solutions. These are of fundamental importance to algebraic geometry.

Many of us who are not experts in pure mathematics know of his famous book *Number Theory: An Approach Through History*. In this he expounds (almost to the layman) number theory—the queen of mathematics—'the oldest, the purest, the loveliest, most elementary and yet most sophisticated field'. His exposition spans 36 centuries of arithmetical work from an old Babylonian tablet datable to Hammurapi to Legendre.

Weil spent two years in India during one of the most turbulent and exciting periods of our history. He managed to see more of India than most Indians do and met many Indians—big and small. He describes the happenings and people with an instinctive feel and understanding never reached before by visitors to India. We reproduce (**page 248**) a few extracts from this book, especially those connected with his Indian sojourn.

Claude Shannon crosses 75

When Claude Shannon, acknowledged as 'the father of information theory', completed 75 years, John Horgan of *Scientific American* wrote a perceptive article in *IEEE Spectrum* (1982, April, p. 72) describing the man and his science. All must read this article to understand the working of this 'irreverant but shy scientist/engineer'. Such is the modesty of this man that he rarely makes an appearance in public. So once recently when he did show up at an International Conference on Information Theory—a field he literally created—the young devotees of this subject were very shocked for they did not know that he was still alive—and they queued up to get the autograph of their fabled hero. 'It was as if Newton had shown up at a physics conference,' one said. He and his theory have received adulation by the shovelful—'The Magna Carta of the communication age'. 'The greatest in the annals of technological thought'. In spite of his receiving National Medal for Science and the Kyoto Prize (the Japanese equivalent of the Nobel prize), he is considered by some to be a crank, and a prankster. This man who caused the communication revolution unshamedly says: 'I have pursued my interest without much regard for financial value or, for that matter, value to the world. I have spent lots of time on totally useless things.'

At the age of 22 he pointed out the relevance of using the little known algebra invented by George Boole in the mid-nineteenth century for use in switching circuits—resulting in intricate circuits being tested mathematically even before they were built! Nowadays Boolean algebra has become essential reading for communication engineers.

His *magnum opus* was created when he was at Bell Telephone Laboratories. He considered how to improve information transmission in a noisy telegraph or telephone line. As a first step he reduced a communication system to its simplest form—a message being represented by one of two states; a binary system of yes or no; on or off, 1 or 0. He then demonstrated that information was a measurable commodity; the amount of information in a given message is determined by the probability that of all the messages that could be sent, *that particular message* could be selected. The overall potential in the system for information he defined as its entropy which is identical to the thermodynamic definition of randomness. Then he showed that every communication line or system has a maximum capacity for transmitting information reliably, and that one can only approach this maximum—the Shannon limit—but one can never reach it. Shannon was, of course, much ahead of his time—the codes needed to reach the Shannon limit were too complicated for vacuum tubes to handle. One had to wait for 25 years when with the advent of high speed integrated circuits, engineers began to exploit Shannon's Information Theory to shape virtually all systems from supercomputers to compact discs.

It is interesting that as with all papers with revolutionary ideas in them, Shannon's too got a negative review when submitted to a journal! It was written in 1941 (when he was 25), but was delayed due to the war and came out in print only in 1948. Benoit Mandelbrot who too brought in revolutionary ideas into 20th century science, sums up about Shannon: 'The great distinction of information theory born fully armed in 1948, from two celebrated papers by Claude Shannon were the novel manner in which it recombined two of the oldest and most fundamental ideas of formalized science, the concept of algorithm and the concept of chance. These two have never been far from each other starting from the games of pure chance of the 17th and 18th centuries continuing through statistical mechanics through 1900 and quantum theory around 1925'.

On **page 253** is an article by Maurice Dodson wherein he discusses many aspects of the historical evolution of the information theory and shows that modern communication systems depend on the equivalence of continuous and digital signals. 'Shannon's sampling theorem is the key to the understanding of this equivalence of the discrete and the analogue. It also lies at the heart of much mathematics and is especially related to the Diophantine approximation. The sampling theorem has therefore links with number theory and the dynamic systems.'

Statistics of language and information theory

Many great names have been associated with the study of statistical laws that seem to exist in language and its structure. Markov (associated with the Markovian process in statistics) in 1913 examined the words in Pushkin's novel *Eugene Onegin* to see whether the statistics of words in it follow a pure chance model.

Shannon's primary motivations in developing the quantitative concept of information was the application to communication (and hence computer language). But the universality of Shannon theory and its relation to thermodynamic entropy lends itself for use in several diverse fields in which language is important.

Some of the linguistic laws were discovered by telegraphists and stenographers whose common aim is to evolve a code by which the machine or hand can express information in as short a time as possible. Many laws, which appear strange to the layman, were discovered independently by Jean Baptiste Estoup (of the Institut de Stenographie (Paris) and George Kingsley Zipf (of Harvard). The most frequently used words in any natural vocabulary are the shortest ones and there are much fewer shorter ones than longer ones (e.g. 25 words could account for nearly 50% of all words in a discourse). Conversely the infrequently used words of a language tend not only to be numerous but also to be long! Further the famous Zipf law of linguistics proposes a constant relationship between the *rank* of a word (r) in a frequency list and the frequency (f) with which it is used in a text, i.e. $r \times f =$ constant.

Mandelbrot (famous for fractals) in his PhD thesis made the first attempt to use Shannon's Information Theory to linguistics. He defined a word as a collection of alphabetic symbols separated by a space. He then found that the brain must have a coding mechanism (*à la* Shannon) for obtaining the Zipf's law.

On page 261 S. Naranan and V. K. Balasubrahmanyan use the word as a primary symbol and show that a natural constraint of the finite vocabulary of a writer (or speaker) leads to a modified version of Zipf's law. This derivation is also based on Shannon's theory. These authors go further and develop models for the length distribution of words, phrases and even sentences. They conclude that the analysis of language seems to provide several parallels to multicomponent physical systems due to the general applicability of thermodynamic-like principles.