FREE OSCILLATIONS OF THE EARTH

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ABSTRACT – *In this article, we review the step-by-step progress made in the recent past towards the solution of the problem of the free oscillations of the Earth. The equations of motion for the toroidal and the spheroidal oscillations of a spherically-symmetric, non-rotating, elastic, isotropic (SNREI) Earth model are given. These equations can be expressed in the form* $dY/dr = A(r)Y$, *where* Y *is a* (6×1) *matrix in the case of the spheroidal oscillations and a* (2×1) *matrix in the case of the toroidal oscillations. The effect of the rotation of the Earth on its free oscillations and the method of calculating the dispersion of surface waves on a sphere are briefly mentioned.*

INTRODUCTION

The seismological data can be analysed and interpreted in terms of three basic concepts. The first is based on the geometrical ray theory of optics. Most of our knowledge about the interior of the Earth has come from the application of the ray-theoretic methods to seismological data. Ray theory is useful for periods shorter than a minute. Travel-times of seismic phases constitute an important component of the ray theory concept.

The second concept is that of surface waves. The study of the dispersion of surface waves helps us in determining the regional structure, e.g. low-velocity zones, oil-bearing structures, etc. The surface wave theory is useful in the period range 10–300 s.

The third concept is that of the normal-mode theory or free oscillations of the Earth. In here, the field is viewed as a sum of standing waves instead of travelling waves. The standing waves can be seen as peaks in the power spectrum (squared-amplitude spectrum) of a seismogram. Theoretical seismograms can now be computed for spherically-symmetric, non-rotating, elastic, isotropic (SNREI) Earth models by mode-summation techniques at low frequencies. At high frequencies, the number of modes required to represent the field becomes so large that the mode-summation technique is not practicable.

The problem of the oscillations of an elastic sphere is an old one. It attracted many classical physicists of the last century, including S. D. Poisson, Lord Kelvin, G. H. Darwin, and H. Lamb. In a classic paper, Lamb[1] discussed the various modes of oscillations of a uniform elastic sphere and calculated the important roots of the frequency equation. He classified the oscillations of a sphere as of the 'first class' and the 'second class'. In an oscillation of the first class, the dilatation and the radial component of the displacement vanish everywhere so that there is no change of shape. In the case of the oscillations of the second class, the radial component of the vorticity vanishes. The oscillations of the first class are now known as toroidal or torsional oscillations and the oscillations of the second class as spheroidal or poloidal oscillations.

The Earth differs from a homogeneous elastic sphere in two important respects: self-gravitation and radial heterogeneity. Gravity does not affect the toroidal oscillations because these are divergence-free and have no radial component of the displacement, so that the density distribution remains unchanged. But gravity plays an important role in the case of the spheroidal oscillations. Bromwich[2] investigated the effect of self-gravitation, simplifying his work by dealing only with an incompressible sphere. Love[3] showed that a self-gravitating sphere of the size and mass of the Earth and of rigidity that of steel has a free period of oscillation of almost exactly one hour for the mode ${}_0S_2$ (the fundamental mode of the spheroidal oscillation of the second order). Obviously, if this mode were to be observed, instruments would be needed that had sensitivity at periods very much longer than the periods of seismic motion then being routinely recorded.

Hugo Benioff was able to record a ground motion of 57 min period for the Kamchatka earthquake of 4 November 1952 on his strain-meter. Benioff[4] attributed it to the mode $_0S_2$ of the oscillations of the Earth. This was the first time that the natural oscillations of the whole Earth had been observed. It was but natural that several seismologists started investigating the problem of the free oscillations of the Earth theoretically and calculating the periods of the normal modes of real Earth models. Jobert[5] applied Rayleigh's principle in calculating the period for the mode $_0T_2$ as 43.54 min for the Bullen B model of the Earth. In the following year, Jobert[6] applied the same method in obtaining the period for the mode $_0S_2$ as about 53 min. Pekeris and Jarosch[7] applied variational methods in calculating the period of the mode $_0S_2$ as 53 min. Soon after Alterman *et al.*[8] integrated the equations of motion numerically and calculated the periods of various normal modes of the Earth. In particular, they showed that the period of the mode $_0S_2$ should be 53.7 min for Bullen B model of the Earth. A wide range of normal modes of the whole Earth were observed by several seismological groups in the U.S.A. for the great Chilean earthquake of 22 May 1960. The agreement between the observed periods and the periods calculated by Alterman *et al.*[8] was excellent. This was the beginning of a new branch of seismology–the low frequency seismology.

Observations of the 1960 Chilean earthquake indicated that the mode $_0S_2$ did not have a single spectral peak, but, instead, was composed of at least two lines with periods 53.1 and 54.7 minutes. Indeed, Pekeris *et al.*[9] and Backus and Gilbert[10] showed theoretically that for the mode $_0S_2$, five spectral lines exist if the rotation of the Earth is taken into account, and calculated their periods.

The work of Alterman *et al.*[8] and their contemporaries did not include the effects of lateral heterogeneity, ellipticity, pre-stress, anisotropy and anelasticity into account. During the last two decades considerable progress has been made in the direction of taking one or more of these features into account (Dahlen[11-14], Dahlen and Sailor[15], Luh[16], Madariaga[17], Smith[18], Takeuchi and Saito[19], Woodhouse[20], Woodhouse and Dahlen[21]). Other important papers on the free oscillations of the Earth published in the recent past include Ben-Menahem *et al.*[22], Dahlen and Smith[23], Derr[24], Gilbert[25,26], Gilbert and Dziewonski[27], Gilbert and MacDonald[28], Landisman *et al.*[29], Phinney and Burridge[30], Saito[31], Singh[32], Singh and Ben-Menahem[33-35] and Wiggins[36].

EQUATIONS OF MOTION

The equations of motion of an SNREI Earth model may be written in the form (Aki and Richards[37], Ben-Menahem and Singh[38], Lapwood and Usami[39])

$$\operatorname{div}\overline{\overline{T}}+\rho[\operatorname{grad}(\Psi-gu_r)+\vec{e}_r g\operatorname{div}\vec{u}]+\vec{f}=\rho\ddot{\vec{u}}, \tag{1}$$

$$\nabla^2\Psi=4\pi G\operatorname{div}(\rho\vec{u}), \tag{2}$$

where

$\vec{u}$ = displacement vector,
$\ddot{\vec{u}}=\partial^2\vec{u}/\partial t^2$,
$\overline{\overline{T}}=\lambda\overline{\overline{I}}\operatorname{div}\vec{u}+\mu(\nabla\vec{u}+\vec{u}\nabla)$ = stress tensor,
λ,μ = Lamé constants,
g = acceleration due to gravity,
G = gravitational constant,
Ψ = perturbation in the gravitational potential,
$\vec{f}$ = body force per unit volume,
ρ = density.

Equations (1) and (2) are to be solved subject to suitable boundary conditions. Thus, $\vec{u}$, $\vec{e}_r\cdot\overline{\overline{T}}$, Ψ and $\partial\Psi/\partial r-4\pi G\rho u_r$ must be continuous everywhere. However, at a boundary between a solid and a fluid or between two fluids, u_r rather than $\vec{u}$ must be continuous.

In the absence of external body force $\vec{f}$, the motion of the Earth consists of a superposition of free simple harmonic oscillations of the form

$$\vec{u}(\vec{r},t)=\vec{u}(\vec{r})\exp(i\omega t).$$

Therefore, the equation which governs the free oscillations of the Earth is obtained from equation (1) by making the substitution $\partial/\partial t\to i\omega$. This yields

$$\mathscr{L}\vec{u}+\rho\omega^2\vec{u}=0, \tag{3}$$

where $\mathscr{L}$ is a self-adjoint linear operator. Equations (2) and (3) together with boundary conditions define an eigenvalue problem. Every model of the Earth will possess an infinite number of eigenfrequencies ω_k ($k=1,2,\ldots$), and a corresponding infinite number of eigenfunctions $\vec{u}_k(\vec{r})$.

As mentioned in the Introduction, the oscillations of the Earth are of two types–toroidal and spheroidal. At high frequencies, the toroidal oscillations correspond to the Love surface waves and *SH* body waves. Toroidal eigenfunctions are of the form

$$\vec{u}(\vec{r})=W(r)[-\vec{e}_r\times\operatorname{grad}Y_{ml}], \tag{4a}$$

where

$$Y_{ml}=P_l^m(\cos\theta)\exp(im\varphi)$$

is a surface spherical harmonic. For every value of l except zero, there is a fundamental toroidal mode $_0T_l$ and an infinite number of overtones $_nT_l$ ($n=1,2,\ldots$). There is no toroidal oscillation corresponding to $l=0$, because the corresponding displacement is identically zero. Also, the mode ${}_0T_1$ does not exist since it corresponds to a rigid rotation. From equation (4a) it is seen that, for toroidal oscillations, $u_r = \text{div}\,\vec{u} = 0$.

At high frequencies, the spheroidal oscillations correspond to the Rayleigh surface waves and $P-SV$ body waves. The spheroidal eigenfunctions are of the form

$$\vec{u}(\vec{r}) = \vec{e}_r\, U(r)\, Y_{ml} + V(r) \text{ grad } Y_{ml}. \quad (4b)$$

The spheroidal modes corresponding to $l=0$ are called radial modes because their associated particle motion is purely radial. For every value of l, there is a fundamental spheroidal mode ${}_0S_l$ and an infinite number of overtones ${}_nS_l$ ($n=1, 2, \ldots$). For $l=1$, the fundamental mode does not exist since it corresponds to a rigid translation. From equation (4b), it can be verified that $(\text{curl}\,\vec{u})_r = 0$. On putting $\Psi(\vec{r}) = P(r)\, Y_{ml}$, and using equations (4a, b), equations (2) and (3) can be transformed into a set of four ordinary differential equations of the second order in U, V, W and P. One of these is an equation in W alone and corresponds to the toroidal oscillations. The remaining three equations are coupled in U, V and P, and correspond to the spheroidal oscillations. These equations are awkward for numerical integration, because one needs to evaluate the derivatives of empirically determined quantities $\lambda(r)$, $\mu(r)$, and $\rho(r)$ in order to obtain the coefficients. This can be formally obviated by converting these equations into an equivalent set of eight linear differential equations of the first order which are free from the derivatives of λ, μ and ρ. Out of these eight equations, two correspond to the toroidal oscillations and the remaining six to the spheroidal oscillations. We can thus express the equations governing the free oscillations of an SNREI Earth model in the form

$$\frac{dY}{dr} = A(r)\, Y, \quad (5)$$

where Y is an $(N \times 1)$ column matrix and $A(r)$ is an $(N \times N)$ square matrix. The number N depends upon the kind of oscillations and the nature of the medium (Table 1).

Table 1

Oscillations	Medium	Gravity	N
Toroidal	Solid	No effect	2
Spheroidal	Solid	Yes	6
Spheroidal	Liquid	Yes	4
Spheroidal	Solid	No	4
Spheroidal	Liquid	No	2
Radial	Solid	Yes	2

For toroidal oscillations, we have

$$\frac{dy_1}{dr} = \frac{y_1}{r} + \frac{y_2}{\mu}, \quad (6a)$$

$$\frac{dy_2}{dr} = \left[(L^2-2)\frac{\mu}{r^2} - \rho\omega^2\right]y_1 - \frac{3}{r}y_2, \quad (6b)$$

where

$$L^2 = l(l+1),\ y_1 = W,\ y_2 = \mu\left(\frac{dW}{dr} - \frac{W}{r}\right).$$

For spheroidal oscillations of a radially heterogeneous, self-gravitating Earth model, the set of equations is

$$\frac{dy_1}{dr} = -\frac{2\lambda}{\delta r}y_1 + \frac{y_2}{\delta} + \frac{\lambda L^2}{\delta r}y_3, \quad (7a)$$

$$\frac{dy_2}{dr} = \left(-\rho\omega^2 - 4\frac{\rho g}{r} + \frac{4\mu\varepsilon}{\delta r^2}\right)y_1 - \frac{4\mu}{\delta r}y_2 + \frac{L^2}{r}\left(\rho g - \frac{2\mu\varepsilon}{\delta r}\right)y_3 + \frac{L^2}{r}y_4 - \rho y_6, \quad (7b)$$

$$\frac{dy_3}{dr} = -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu}, \quad (7c)$$

$$\frac{dy_4}{dr} = \left(\frac{\rho g}{r} - \frac{2\mu\varepsilon}{\delta r^2}\right)y_1 - \frac{\lambda}{\delta r}y_2 + \left[-\rho\omega^2 + \{(1+\varepsilon/\delta)L^2 - 2\}\frac{\mu}{r^2}\right]y_3 - \frac{3}{r}y_4 - \frac{\rho}{r}y_5, \quad (7d)$$

$$\frac{dy_5}{dr} = 4\pi G\rho y_1 + y_6, \quad (7e)$$

$$\frac{dy_6}{dr} = -4\pi\frac{L^2}{r}G\rho y_3 + \frac{L^2}{r^2}y_5 - \frac{2}{r}y_6, \quad (7f)$$

where

$$\delta = \lambda + 2\mu,\ \varepsilon = 3\lambda + 2\mu,$$

$$y_1 = U,\ y_2 = \delta\frac{dU}{dr} + \frac{2\lambda}{r}U - \lambda\frac{L^2}{r}V,$$

$$y_3 = V,\ y_4 = \mu\left[\frac{1}{r}(U-V) + \frac{dV}{dr}\right],$$

$$y_5 = P,\ y_6 = \frac{dP}{dr} - 4\pi G\rho U.$$

In the case of the liquid core, $\mu = y_4 = 0$ and equation (7d) yields

$$y_3 = \frac{1}{r\omega^2}\left(g y_1 - \frac{y_2}{\rho} - y_5\right). \quad (8)$$

Equations satisfied by y_1, y_2, y_5, y_6 are obtained from equations (7a, b, e, f) on putting $\mu = y_4 = 0$ and substituting the above value of y_3.

The equations for the spheroidal oscillations of a nongravitating Earth model can be obtained from equations (7a, b, c, d) on putting $g = y_5 = y_6 = 0$. If we further put $\mu = y_4 = 0$ and $y_3 = -y_2/(\rho r \omega^2)$ obtained from equation (8), we get the following equations for the spheroidal oscillations of a non-gravitating liquid core:

$$\frac{dy_1}{dr} = -\frac{2}{r} y_1 + \left(\frac{1}{\lambda} - \frac{L^2}{\rho \omega^2 r^2}\right) y_2,$$

$$\frac{dy_2}{dr} = -\rho \omega^2 y_1 .$$

In the case of the radial oscillations, $l = y_3 = 0$. Equation (7f) then yields

$$\frac{dy_6}{dr} = \frac{-2}{r} y_6$$

which integrates to $y_6 = C/r^2$. In order to avoid the infinity at the centre, we take $C = 0$. Equations (7a, b, c, e) now yield

$$\frac{dy_1}{dr} = -\frac{2\lambda}{\delta r} y_1 + \frac{y_2}{\delta},$$

$$\frac{dy_2}{dr} = \left(-\rho\omega^2 - 4\frac{\rho g}{r} + \frac{4\mu\varepsilon}{\delta r^2}\right) y_1 - \frac{4\mu}{\delta r} y_2,$$

$$y_4 = \frac{\mu}{r} y_1,$$

$$\frac{dy_5}{dr} = 4\pi G \rho y_1$$

Equation (5) subject to an appropriate set of homogeneous boundary conditions (Ben-Menahem and Singh[38]) can be integrated by the Runge-Kutta method. For a given SNREI Earth model $\lambda(r)$, $\mu(r)$ and $\rho(r)$ and given l, there is a discrete set of values of ω for which the boundary conditions are satisfied. We denote this set by ω_k (eigenfrequencies) and the corresponding displacements by $\vec{u}_k$. The eigenperiods are given by $T_k = 2\pi/\omega_k$. The index k signifies the normal-mode. Each k stands for a triplet (l, m, n) of numbers; l being the colatitudinal mode number $(l \geq 0)$, m the azimuthal model number $(-l \leq m \leq l)$ and n the radial mode number $(n \geq 0)$. The numbers l, m, n describe the manner in which the displacement field depends upon the colatitude, the azimuth, and the radial distance, respectively. However, because of the spherical symmetry, the eigenfrequencies are degenerate in m (i.e., ω_k are independent of m). This degeneracy is removed when the rotation of the Earth, its ellipticity or lateral heterogeneity is taken into account.

Table 2 Fundamental mode ($n = 0$) eigenperiods, in seconds, for the Jeffreys-Bullen A' model of the Earth.

	Toroidal		Spheroidal	
l	Calculated	Observed	Calculated	Observed
2	2610	2579	3206	3232
3	1690	1707	2116	2134
4	1293	1306	1531	1546
5	1067	1076	1179	1189
6	918	926	955	964

Table 2 compares the eigenperiods computed for the Jeffreys-Bullen A' Earth model (Ben-Menahem and Singh[38]) *with the observed periods.*

Effect of the rotation of the Earth

As mentioned earlier, the eigenfrequencies of a spherically symmetric, non-rotating Earth model are degenerate in the azimuthal order number m. The introduction of the rotation removes this degeneracy. The effect of the diurnal rotation of the Earth can be calculated by carrying out a first-order perturbation calculation. Let $\vec{\Omega}$ denote the uniform angular velocity of the Earth about its centre. Assume that the observer is referred to a noninertial frame, which, for all times, maintains a state of uniform rotation with angular velocity $\vec{\Omega}$. Let (x_1, x_2, x_3) be a Cartesian coordinate system in this uniformly rotating frame of reference, let the origin of the system coincide with the centre of the Earth, and let $\vec{e}_3$ be aligned along the axis of rotation, so that $\vec{\Omega} = \Omega \vec{e}_3$.

The rotation of the Earth introduces two body forces. One of them is the centrifugal force $r \sin\theta\ \Omega^2$ perpendicular to and away from the axis of rotation. The other force introduced by the rotation is the Coriolis force, which, per unit mass, is equal to $2\Omega\, \dot{\vec{u}} \times \vec{e}_3$.

Assuming $\Omega/{}_n\omega_l \ll 1$ (the highest value of $\Omega/{}_n\omega_l \simeq 1/27$) and carrying out a first order perturbation calculation, it can be shown that (Ben-Menahem and Singh[38])

$${}_n\omega_l^m = {}_n\omega_l + m\tau\Omega, \quad -l \leq m \leq l, \tag{9}$$

where

${}_n\omega_l$ = eigenfrequencies of a non-rotating Earth model,

${}_n\omega_l^m$ = eigenfrequencies of the corresponding rotating Earth model,

and the splitting parameter

$$\tau = \frac{1}{l(l+1)} \tag{10}$$

for toroidal oscillations, and

$$\tau = \frac{\int_0^a (2U+V)\, V\rho r^2 \mathrm{d}r}{\int_0^a [U^2 + l(l+1)V^2]\rho r^2 \mathrm{d}r} \tag{11}$$

for spheroidal oscillations.

Therefore, for both toroidal and spheroidal oscillations, the degenerate eigenfrequency ${}_n\omega_l$ is resolved by a slow rotation of the Earth into $(2l+1)$ frequencies ${}_n\omega_l^m (-l \le m \le l)$. The set of $(2l+1)$ spectral lines for a given (l, n) is called a mode multiplet and each member of this set is called a singlet. For the toroidal oscillations the splitting parameter $[l(l+1)]^{-1}$ does not depend upon the Earth model. However, in the case of the spheroidal oscillations, τ depends upon the Earth model. The splitting of the terrestrial spectral lines is the elastodynamic analogy of the splitting of atomic spectral lines by a magnetic field discovered by Zeeman in 1896.

The ellipticity of the Earth also removes the degeneracy giving (Dahlen[11])

$${}_n\omega_l^m = {}_n\omega_l\, [l + e(b + m^2 c)] \tag{12}$$

to first order in ellipticity e, where $b = b(n, l)$, $c = c(n, l)$. Thus the ellipticity splits the degenerate eigenfrequency ${}_n\omega_l$ into $(l+1)$ lines ${}_n\omega_l^m (0 \le m \le l)$.

JEANS' FORMULA

The complete seismic field induced by a point source in an SNREI Earth model can be expressed as an infinite sum of normal modes. However, we know from seismogram analysis that most of the recorded Earth motion can be explained in terms of propagating waves. Therefore, there must exist a relationship between these two seemingly different aspects of seismic wave motion.

Consider a general term in the normal-mode solution expressed by equations (4a, b). The factor of this term which depends upon the time and the colatitude is derivable from

$$P_l^m(\cos\theta)\exp(i\omega t), \tag{13}$$

where $\omega = {}_n\omega_l$ is an eigenfrequency. Replacing the Legendre function by its asymptotic approximation for large values of l, this factor becomes

$$(-l)^m \left(\frac{1}{2\pi l \sin\theta}\right)^{1/2} \left[\exp\left\{i\left[\omega t - (l+1/2)\theta + \frac{\pi}{4} - \frac{m\pi}{2}\right]\right\} + \exp\left\{i\left[\omega t + (l+1/2)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right]\right\}\right] \tag{14}$$

The first-term describes a wave motion of frequency ω diverging from the pole $\theta = 0$, whereas the second term describes a wave motion diverging from the antipode $\theta = \pi$.

In cylindrical coordinates, a wave diverging from the axis may be expressed in the form

$$\exp[i(\omega t - k\Delta)], \tag{15}$$

where k is the wave number. Putting $\Delta = a\theta$ and comparing (14) and (15), we get Jeans' formula:

$$l + \tfrac{1}{2} = ka \quad (l \gg 1). \tag{16}$$

This formula tells us that, if l is large, every mode of oscillation can be interpreted as a travelling wave whose parameters are functions of l and n and are given by

$$T = \frac{2\pi}{\omega}, c = \frac{a\omega}{l+\frac{1}{2}}, \lambda = \frac{2\pi a}{l+\frac{1}{2}}, \tag{17}$$

where T is the period of the oscillation, c is the phase velocity, and λ is the associated wave length. These relations are found to yield a good approximation for $l \ge 7$. The group velocity can be calculated from the relation

$$U_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = a\frac{\mathrm{d}\omega}{\mathrm{d}l} = c + (l + \tfrac{1}{2})\frac{\mathrm{d}c}{\mathrm{d}l}. \tag{18}$$

In practice, the derivative with respect to l is obtained by finite differencing. Clearly, this formula is not valid for small values of l.

The phase velocity data calculated by means of the Jeans' formula for the period range 300–3000 s can be combined with the surface wave dispersion data at shorter periods, and the entire spectrum can be used to find the appropriate model of elastic properties of the Earth.

SURFACE WAVES ON A SPHERE

Surface waves are introduced for models with plane-parallel boundaries. The real Earth differs from a vertically heterogeneous half-space model in three important respects: sphericity, gravity, and liquid core. The flat-Earth model gives a good approximation for $T < 50$s. In the period range $50\text{s} < T < 300\text{s}$, a flat-Earth model with suitable Earth-flattening approximation serves our purpose[40,41,42]. Propagating waves in the period range 300–600s, are sometimes observed in seismograms of major earthquakes. The corresponding normal modes have significant amplitudes not only in the crust and mantle but in the core as well. Their motion is governed by the elastic restoring forces as well as by gravitational forces. In the range $300\text{s} < T < 400\text{s}$, the normal-mode amplitudes in the core become negligibly small so that it is sufficient to carry out the

TABLE 3

Period(s)	l	Method	Core included	Gravity included
$T<50$	$l>200$	Flat Earth	No	No
$50<T<300$	$200>l>25$	Earth-flattening approximation	No	No
$300<T<400$	$25>l>17$	Restricted normal mode	No	Yes
$T>400$	$l<17$	Complete normal mode	Yes	Yes

integration from the base of the mantle. In this restricted normal-mode method, we assume that all the components of the displacement vector and the radial stress vector vanish at the core-mantle boundary. The ranges of applicability of the various methods are given in *Table 3*.

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ACKNOWLEDGEMENT: The author is thankful to the University Grants Commission, New Delhi for financial support through the major research project "Generation and Propagation of Seismic Waves–a Theoretical Study".