development costs and easily place a mechanism that has proven successful elsewhere, in place in a short time.

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REVIEW ARTICLE

Quantum mechanics and statistical mechanics of anyons

Avinash Khare
Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India

I discuss in some detail the quantum mechanics and statistical mechanics of anyons which are objects in two space dimensions obeying statistics which is interpolating between Fermi–Dirac and Bose–Einstein statistics. In particular I discuss the quantum spectrum of two and three anyons experiencing harmonic oscillator potential. Using these results I discuss the computation of the second virial coefficient of an anyon gas. Approximate results for the third virial coefficient are also given. Some discussion is also given about the possible relevance of anyons in condensed matter physics. Finally it is pointed out that the charged vortices in the abelian Higgs model with the Chern–Simons term provide a concrete model for charged anyons in relativistic field theory.

MANY of us have wondered some time or other if one can have nontrivial science and technology in two space dimensions; but the general feeling is that two-space dimensions do not offer enough scope for it. To my knowledge this whole question was first addressed in 1884 by E. A. Abbot in his satirical novel *Flatland*. The first serious book on this topic appeared in 1907 entitled *An Episode of Flatland*. In this book C. H. Hinton[1] offered the first glimpses of the possible science and technology in the flatland. In 1979 A. K. Dewdney[2] published a 97-page book which contains in detail the laws of physics, chemistry, astronomy and biology in the flatland. However, all these people missed one important case where physical laws are much more complex, nontrivial and hence interesting in the flatland than in our three-dimensional world. I am referring to the case of quantum statistics. In particular, in the last 14 years it has been realized that whereas in three and higher space dimensions all particles must be bosons or fermions (i.e. they must have spin of $n\hbar$ or $(2n+1)\ \hbar/2$ with $n = 0,1,2,\ldots$ and must obey Bose–Einstein or Fermi–Dirac statistics respectively), in two space dimensions the particles can have any fractional spin and can satisfy any statistics which is interpolating between Bose and Fermi statistics—hence the name 'anyons' for such particles. In other words, if we take one anyon slowly around the other then the phase acquired can in general be $\exp(i\theta)$. If $\theta = 0$ or π (modulo 2π) then the particles are bosons or fermions respectively while if $0 < \Theta < \pi$ then the particles are termed as anyons. In this article I point out that the charged vortex solutions[3] in the abelian Higgs model with the Chern–Simons term provide a concrete model for charged anyons in relativistic field theory.

Why exotic spin and statistics in two dimensional?

It is not very difficult to understand as to why angular momentum need not be quantized in two space dimensions. The point is that spin in two dimensions differs fundamentally from spin in higher dimensions. This is because whereas the angular momentum algebra is noncommutative in three and higher space dimensions

$$[J_i, J_j] = 2i\varepsilon_{ijk}J_k;\ i, j, k = 1, 2, 3 \tag{1}$$

it is a trivial commutative algebra in two space dimensions since only one generator (say J_z) is available. As a result, there is no analogue of the quantization of angular momentum, which arises in three and higher dimensions from the nonlinearity of the commutation relations associated with the nonabelian rotation group. Thus the angular momentum of particle states need not be $n\hbar$ or $(2n+1)\,\hbar/2$ but could take any arbitrary value. Now in relativistic quantum field theory, there is a deep and fundamental connection between spin and statistics. Particles with half-integral spin are fermions, those with integer spin are bosons. This immediately suggests that in two space dimensions the particles may exhibit exotic statistics. In one of the most remarkable and clearly written papers Leinaas and Myrheim[4] showed that this expectation is indeed realized. The two key arguments[5] in the proof are (i) indistinguishability of identical particles in quantum mechanics, and (ii) Feynman's path-integral approach to quantum mechanics. The principle of indistinguishability of identical particles is even older than quantum mechanics. It was recognized by Gibbs long before and is in fact at the root of his entropy paradox in classical statistical mechanics. The key point of the whole discussion is that if the coordinate space of a one-particle system is X (typically X is the d-dimensional Euclidean space R^d) then the true configuration space of the N-particle system is *not* X^N but X^N/S_N, which is obtained from X^N by dividing out by the action of the symmetric group S_N of N identical particles. In this way it was shown that if one anyon is slowly taken around the other in anticlockwise direction then the phase acquired is $e^{i\theta}$ while it is $e^{-i\theta}$ if it is taken around in clockwise direction with $0 \le \Theta \le \pi$; Θ being a continuous parameter. Several conclusions follow from here. Some of them are: (i) The anyons must necessarily violate parity (P) and time reversal (T) symmetries (if $\Theta \neq 0, \pi$). This is because the counterclockwise windings are related to the clockwise winding by mirror reflection (i.e. parity). Similarly, if counterclockwise windings are reversed in time, they will look like clockwise windings. Thus P and T violation offers a unique test of theories with anyons. For example, in the last few years there have been suggestions[6] that anyons may provide mechanism for high-T_c superconductivity. If true then the *high*-T_c materials must exhibit P and T violation. While the experiments are inconclusive at this time[7], the general feeling is that anyons may not provide the mechanism for *high*-T_c superconductivity; (ii) Anyons are sort of in between bosons and fermions, i.e. the repulsion between two anyons monotonically increases as Θ goes from 0 to π with there being no repulsion between two bosons. As a result one finds that the trajectories of two anyons cannot cross each other and one can in principle distinguish crossing 'in front' from crossing 'behind'. (iii) The technical name for the group structure of the paths for anyons is braid group—a name which is both apt and picturesque! (iv) How does one braid a third particle trajectory into the two? It can be done in a trivial way or in a nontrivial way as shown in Figure 1. Whereas in the first case one has the phase factor of $e^{i\Theta}$ in the second case the appropriate phase factor is $e^{3i\Theta}$. This is bad news because it means that unlike two bosons or two fermions, the phase due to the exchange of two identical anyons in two dimensions depends, in principle, on the position of all the

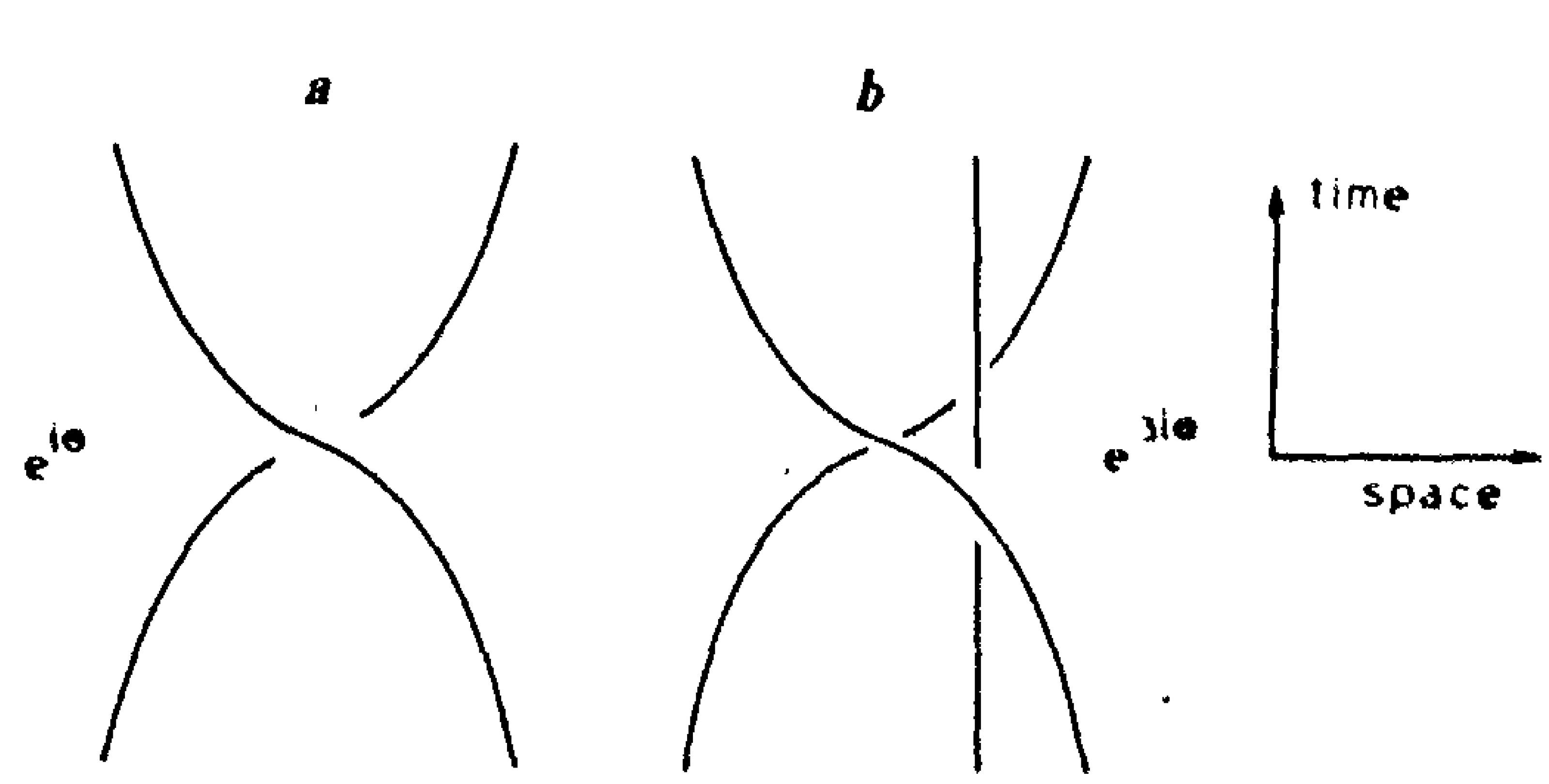

Figure 1. The three-anyon trajectories, (*a*) third 'passive' bystander and (*b*) 'active' braiding which cannot be undone.

particles. This fact make three- and multi-anyon problems highly nontrivial and that is why till today these problems have still remained essentially unsolved.

Possible relevance to the real world

Is our discussion merely of academic interest? The answer to the question is no. It turns out that there are many condensed matter systems that are essentially planer. The point is that in all these cases the states of the motion in the transverse direction are quantized, i.e. it takes a finite amount of energy to excite them. Thus if we consider such systems at sufficiently low temperature so that the energy to excite them is not available then the systems are essentially planer. Few such examples are quantum Hall effect, surface layer studies and copper-oxide materials. Of course, even then at the most basic level the fundamental particles are certainly required to be fermions or bosons. However, the most direct and appropriate discussion of the low energy behaviour of a material is usually in terms of quasi-particles. One can hope that at least in some of these cases the quasi-particles could be anyons. This hope has in fact already been realized in the case of the fractionally quantized Hall effect. In this case the best explanation offered so far is by Laughlin[8], and, according to him, the quasi-particles responsible are charged vortices, i.e. charged anyons.

Another reason why I believe that anyons would have relevance to the real world is because of the unwritten first law of physics which states that 'anything that is not forbidden is compulsory!' In a sense anyons represent a challenge to all those people who think that they know quantum mechanics and statistical mechanics and that they could have contributed to the development of these two subjects in the thirtees if only they had been born 50 years earlier!!

Models for anyons

There are two popular models of anyon depending on if it is a point particle or an extended object. The first model is due to Wilczek[9]. In this picture one can look upon anyon as a point particle with thin flux tube attached at the site of each anyon. As a result anyons carry both magnetic flux and electric charge i.e. they are charged vortices. With this picture one can look upon anyon as boson (or fermion) with statistical interaction given by the Lagrangian $\mathscr{L}_{st} = (\hbar\Theta/\pi)d\phi/dt$, where Θ is the statistical parameter $(0 \leq \Theta \leq \pi)$ and ϕ is the relative angle between two anyons. This can be easily generalized to the case of n anyons. Since this Lagrangian is a total time derivative hence it does not contribute to the equations of motion so that it is purely a quantum mechanical effect. This Lagrangian can be derived from the topological Chern–Simons Lagrangian.

Anyons as charged vortices. In 1986 Samir Paul and myself[3] at Bhubaneswar showed that the abelian Higgs model with the Chern–Simons term in two space and one time dimensions admit charged vortex solutions[3] of finite energy, finite quantized flux, charge and angular momentum which is, in general, fractional. It strongly suggested that these objects could be charged anyons. This was subsequently confirmed by Frohlich and Marchenti[10] using axiomatic quantum field theory.

Quantum mechanics of anyons

Now that the new quantum statistics is possible in flatland, à la the Bose and Fermi case, one would like to study the properties of an ideal gas of anyons. In particular one would like to know the partition function, the momentum distribution function, etc. of an ideal anyon gas. This would be a sort of bench-mark study which is necessary so that in realistic calculations one can have some idea about the validity of the various approximations made. Unfortunately, it turns out that only the two-anyon quantum-mechanical problem is exactly solvable while three- and multi-anyon problems are still unsolved, presumably because of the nontrivial braiding effects. As a result only the second virial coefficient of an ideal anyon gas can be computed.

Let me first discuss a few exact results about two-anyon quantum mechanics and then discuss the latest situation about the three- and multi-anyon problems.

The Lagrangian for two noninteracting anyons (i.e. two bosons interacting via the statistical interaction) is given by

$$\mathscr{L} = \frac{\mu}{2}(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) + \frac{\hbar\theta}{\pi}\dot{\phi}. \tag{2}$$

The corresponding Hamiltonian can be separated into the centre of mass and the relative Hamiltonians. In particular one can show that

$$\mathscr{H}_{\mathrm{rel}} = \frac{P_r^2}{\mu} + \frac{(P_\phi + \frac{\hbar\theta}{\pi})^2}{\mu r^2} \tag{3a}$$

$$\text{where, } P_\phi = \frac{\mu}{2} r^2 \dot{\phi} + \frac{\hbar\theta}{\pi}. \tag{3b}$$

This Hamiltonian has only a continuous spectrum. To see the effect of anyons we add harmonic oscillator potential $(V(r) = (\mu/4)\omega^2 r^2)$ between the two anyons. In that case the Schrodinger equation can be exactly solved and one can show that the bound state spectrum is given by[4]

$$E_{n_r;m}(\theta) = (2n_r + |m + \frac{\theta}{\pi}| + 1)\,\hbar\omega, \quad (4)$$

where the radial quantum number $n_r = 0, 1, 2, \ldots$, the azimuthal quantum number m takes the values $m = 0, \pm 2, \pm 4, \ldots$, while Θ interpolates between 0 and π (modulo 2π) with $\Theta = 0$ corresponding to bosons and $\Theta = \pi$ to fermions. The spectrum $E(\Theta)$ has been plotted in Figure 2 as a function of Θ from where we notice that the spectrum is in general not equispaced unless $\Theta = 0$, π or $\pi/2$ i.e. unless the particles are bosons, fermions or half-fermions (also known as semions). This is a very important result because it means that unlike fermions and bosons, the energy levels of the noninteracting two-anyon system are not simply related to that of a single anyon. That is why one is not able to write down the n-anyon wave function in terms of the single-particle wave function.

We have also studied[11] the problem of the scattering of charged anyons and we find that the Mott scattering cross section between two anyons shows a marked asymmetry between the forward and the backward angles and further there is a dip in the differential cross section. In the case of the attractive Coulomb problem we show that the energy levels cross[5,11] with all the crossings occurring at the semion value of $\Theta = \pi/2$.

Let us now discuss the problem of three anyons in an oscillator potential. Till today this problem has remained essentially unsolved. Wu[12] was the first one to attack this problem and he was able to obtain exact eigenvalues and eigenfunctions of some of the states that include the ground state of the three-bosons but not the ground state of the three-fermions. In particular he showed that the exact expression for the three-anyon bound state (including the three-boson ground state) energy is given by

$$E = (2 + 3\alpha)\,\hbar\omega. \quad (5)$$

Note that whereas for $\alpha (\equiv \Theta/\pi) = 0$ this gives the exact three-boson ground state, for $\alpha = 1$ it does not correspond to the three-fermion ground state (which is at $4\,\hbar\omega$) but to that of an excited state. Thus whereas Wu's exact eigenfunction corresponds to that of ground state for $\alpha < \alpha^*$, for $\alpha > \alpha^*$ it corresponds to that of excited state. Recently, the three-anyon ground (as well as excited) state energy has also been computed to $0(\alpha)$ in perturbation theory around the bosonic end[13] and it has been shown that it is identical with Wu's exact expression[12] as given by eq. (5). This means that the perturbation theory breaks down beyond $\alpha = \alpha^*$. What is the value of α^*? We have recently estimated α^* by computing the three-anyon ground state energy to $0(\alpha^2)$ in perturbation theory around the fermionic end[14] (note that the $0(\alpha)$ correction is zero in this basis) and it has been shown that

$$E = [4 + 1.3(1-\alpha)^2]\,\hbar\omega + \ldots. \quad (6)$$

The two energies as given by eqs. (5) and (6) are plotted in Figure 3. Here the two curves cross each other at $\alpha = 0.7$, thereby indicating that in case there are no further crossings then $\alpha^* = 0.7$, i.e. the bosonic (fermionic) perturbation theory, breaks down for $\alpha > \alpha^*$ ($\alpha < \alpha^*$).

The fact that the three-anyon ground state energy is not maximum for $\alpha = 1$ (three-fermion) but is maximum

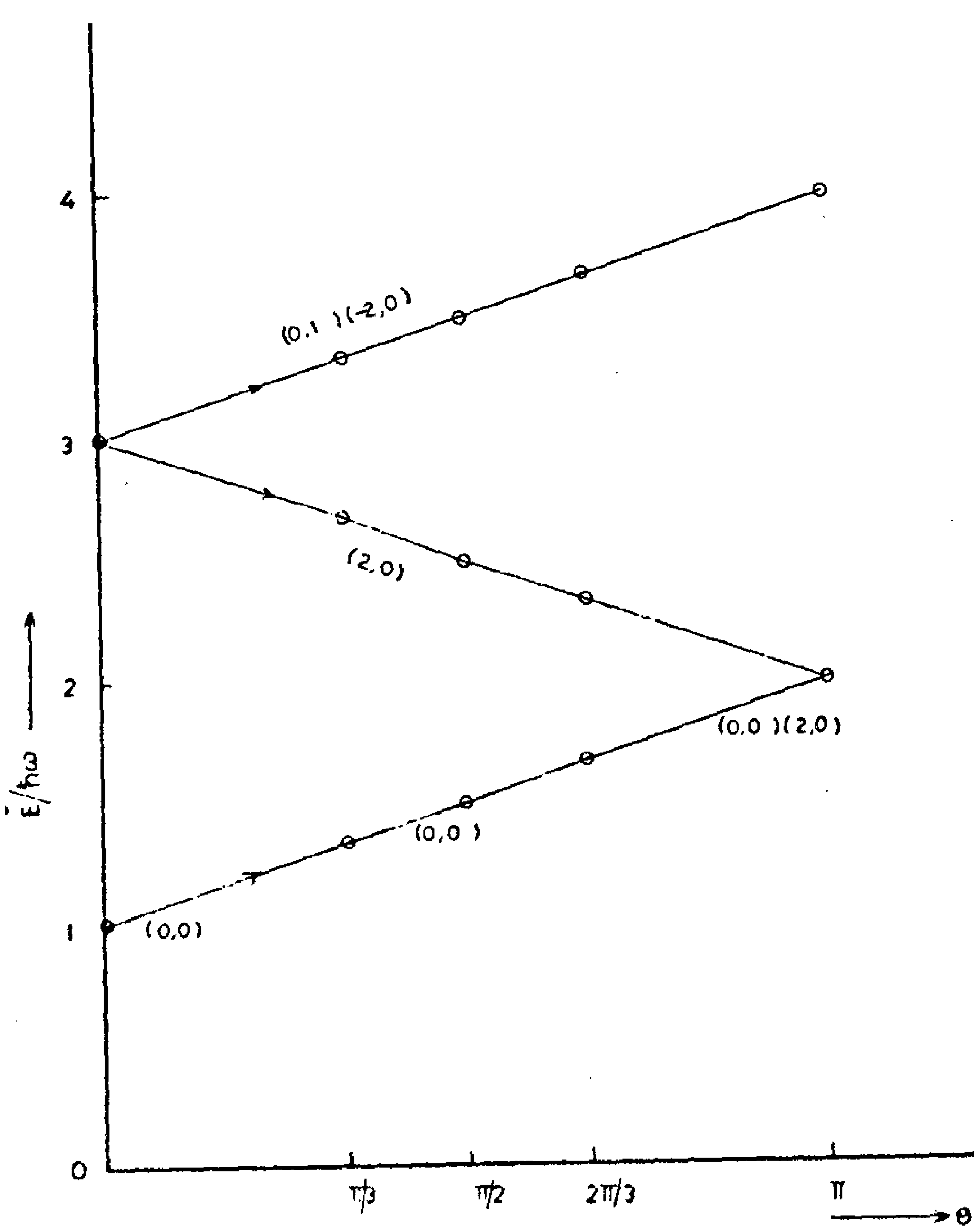

Figure 2. Two-anyon bound state spectrum in oscillator potential.

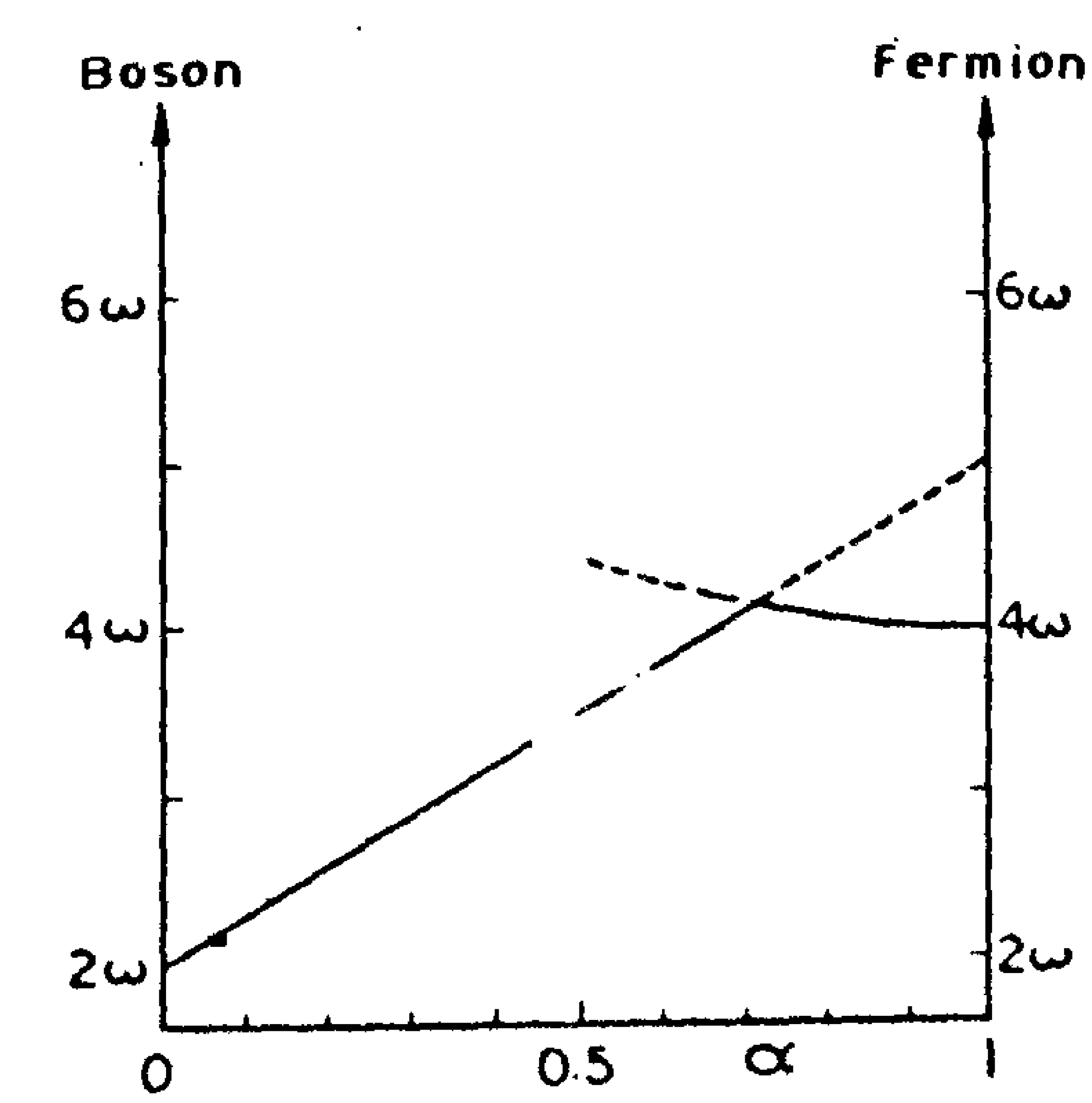

Figure 3. Three-anyon ground state energy around the fermionic and the bosonic basis.

at $\alpha = 0.7$ is somewhat surprising. This is because since the repulsion between two anyons is maximum for fermions ($\alpha = 1$) hence one would have naively expected that the three-anyon ground state energy should be maximum at $\alpha = 1$. I believe that this unexpected result is because of the nontrivial braiding effect; a deeper understanding is however still lacking.

Recently Wu's exact results for three-anyons have been extended to n-anyons in oscillator potential and exact eigenfunctions and eigenvalues have been written down[15] for some of the states. Besides, the low-lying three-anyon spectrum has been numerically calculated[16].

Statistical mechanics of anyons

We would ideally like to have an expression for the partition function of a noninteracting anyon gas but since three and higher anyon quantum mechanical problems are still unsolved, hence the best that one can do is to use the exact results for the two anyon problem and calculate the second virial coefficient of a dilute anyon gas.

Second virial coefficient. The equation of state for a real gas expanded in powers of density ρ reads

$$P V = \frac{N}{\beta}(1 + a_2 \rho + a_3 \rho^2 + \ldots), \quad \beta = \frac{1}{KT}, \tag{7}$$

where a_n represents the nth virial coefficient. For an ideal Bose or Fermi gas in two dimensions it is well known that ($m = 1, 2, \ldots$)

$$a_2^{\mathrm{B}} = -a_2^{\mathrm{F}} = -\frac{\pi \beta \hbar^2}{2m} \equiv -\lambda_T^2,$$

$$a_{2m+2}^{\mathrm{B}} = a_{2m+2}^{\mathrm{F}} = 0, \; a_{2m+1}^{\mathrm{B}} = a_{2m+1}^{\mathrm{F}} \neq 0. \tag{8}$$

One can compute a_2 for anyons[17] by adding a harmonic potential between the anyons so that the spectrum is purely discrete and given by eq. (4). In particular, $a_2(\alpha, T)$ is then given by (note $\alpha = \Theta/\pi$).

$$a_2(\alpha, T) = -\lambda_T^2 \left[1 + \lim_{\omega \to 0} 8 \{ Z^{(2)}(\alpha, \omega) - Z^{(2)}(\alpha, 0) \} \right], \tag{9}$$

where $Z^{(2)}(\alpha, \omega) = \mathrm{Tr}\, e^{-\beta \mathscr{H}_2}$ with $\mathscr{H}_2$ being the two-particle Hamiltonian. Using the spectrum as given by eq. (4) one can then show that

$$a_2(\alpha, T) = -\lambda_T^2 \left[1 + 2\alpha^2 - 4\alpha \right]_{\text{periodic}}, \tag{10}$$

where the subscript indicates that we are to extend these results for $|\alpha| > 1$ in a periodic fashion. From eq. (10) it is easily seen that $a_2(\alpha, T)$ has cusps at Bose values of $\alpha = 2n$. As expected at $\alpha = 1$ we get back the exact second virial coefficient for fermions from here. Instead of using harmonic confinement, one could also have introduced a circular box of radius R so that the spectrum is purely discrete and finally considered the limit $R \to \infty$. One again obtains the same answer[18].

Can one extend this method and calculate a_3 and higher virial coefficient of anyons? The answer is no since we do not know the full three-anyon spectrum. In the absence of the exact solution, one has to take recourse to approximate methods. Using the three-anyon spectrum to $0(\alpha)$ it has been shown[19] that to this order $a_3(\alpha)$ (and even $a_n(\alpha)$; $n \geq 3$) do not have any cusps at Bose values of $\alpha = 2n$.

We have on the other hand used the semiclassical approximation[20] to calculate $a_3(\alpha)$. The advantage of this method is that it does not require a knowledge of the quantum spectrum and is expected to yield accurate results at high temperature. To test the accuracy of this method we first calculated[21] the second virial coefficient by using H_{rel} as given by eq. (2). Now the classical partition function of the system is given by

$$Z_{\mathrm{cl}}^{(2)}(\alpha) = \int_0^{\infty} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}p_r \int_0^{2\pi} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}p_\phi \; \exp\left[-\beta \left(\frac{p_r^2}{\mu} + \frac{(p_\phi + \hbar\alpha)^2}{\mu r^2} \right) \right], \tag{11}$$

The quantum effect is unravelled by recognizing that $p_\phi = \hbar m$ and one replaces $\int \mathrm{d}p_\phi$ by $\hbar \Sigma m$, where $m = 0, \pm 2, \pm 4, \ldots$. In this way we showed[21] that the semiclassical approximation reproduces the exact (quantum) second virial coefficient for anyons. This is quite remarkable but not that surprising if one recalls the work of Comtet and Ouvry[22] who have shown that the second virial coefficient for anyons is related to chiral anomaly and it is well known that the semiclassical approximation is exact for the chiral anomaly. Armed with this success, we then undertook the tedious job of estimating the third virial coefficient for an anyon gas. Unfortunately we were not able to isolate the fermion and boson contributions from the mixed symmetry contributions. We therefore summed over all possible values of the angular momenta. Very accurate numerical calculations showed that the third anyon effectively decouples (in the area) and does not contribute directly to the interacting part of the partition function when the sum over the mixed symmetry states is also included. From the calculations, we obtained[21]

$$a_3(\alpha) = \lambda_T^4 \left[\frac{1}{36} + 4\alpha^2 (1 - |\alpha|)^2 \right]. \tag{12}$$

It is interesting to note that even though approximate (mixed symmetry contributions having been included), this $a_3(\alpha)$ is nevertheless consistent with the two known exact results, i.e. $a_3(\alpha)$ does not have contribution of $0(\alpha)$ and further $a_3^F \equiv a_3(\alpha=1) = \lambda_T^4/36 = a_3^B$.

Concluding remarks

The partition function of a noninteracting anyon gas still remains one of the most challenging, unsolved problems in statistical mechanics mainly because of the nontrivial braiding effects. Unless this problem is solved we would not know if the realistic approximate calculations are reliable or not. I hope I have been able to convince you that the physical laws in 'flatland' could be even more complex, nontrivial and hence interesting than in our real world!

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RESEARCH COMMUNICATIONS

Transfer functions related to zeroth order modified Bessel equation

G. C. Purohit* and B. D. Indu

Crystal Impurity Division, Department of Physics, H. N. B. Garhwal University, P. O. Box 21, Srinagar 246 174, India
*Department of Mathematics, Government Intermediate College, Rudraprayag 246 171, India

We obtain the solution of the zeroth-order modified Bessel equation with the help of double-point space-dynamic Green's functions. The expressions showing the correspondence between transfer functions and Ber, Bei, Ker, Kei functions are derived with the help of this method. We discuss the application of the solution in the case of the distribution of alternating currents in wires.

THE double-point transfer function known as Green's function is the most powerful tool for solving differential equations[1-4]. Here we present the double-point transfer function solution for the zeroth-order modified Bessel equation

$$x\,D^2 y + D y - i p^2 x y = 0, \tag{1}$$

where $D = (d/dx)$. The traditional solution of this equation is given by[5-8],

$$y = A\,I_0(px\sqrt{i}) + B\,K_0(px\sqrt{i}), \tag{2}$$

where the complex functions $I_0(px\sqrt{i})$ and $K_0(px\sqrt{i})$ can be expressed in terms of Bessel real.(Ber), Bessel imaginary (Bei), Ker and Kei functions in the form

$$I_0(px\sqrt{i}) = \mathrm{Ber}(px) + i\,\mathrm{Bei}(px), \tag{3a}$$

$$K_0(px\sqrt{i}) = \mathrm{Ker}(px) + i\,\mathrm{Kei}(px). \tag{3b}$$

Let us consider that there is a vanishing force[9] which governs equation (1) in the form

$$D_1^2 y + \eta^{-1} D_1 y - y = -f_0(\eta, \eta_0). \tag{4}$$

In this equation we have taken $pxi = \eta$ and $D_1 = d/d\eta$. The force $f_0(\eta, \eta_0)$ is known as self-destroying force of