Spin–statistics connection

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The connection between spin and statistics of particles and systems can be established on topological arguments if the notion of charge conjugates and the usual properties of pair creation and annihilation are assumed. On compact two-dimensional surfaces, there are new possibilities of fractional non-abelian statistics[1].

Massive particles or systems are known to have integer or half odd integer spin values and are known to satisfy either Bose–Einstein or Fermi–Dirac statistics. Consequently, two identical particles are described by a wave function with

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1),$$

where symmetric wave function is associated with bosons and antisymmetric ones with fermions. Indeed, for N identical particles this is generalized to

$$\psi(P(x_1, x_2, \ldots, x_n)) = (-1)^p \psi(x_1, x_2, \ldots, x_n),$$

where $p=0$ for all permutations $P\{x_i\}$, goes with bosons and $p=1$ for odd permutation $P\{x_i\}$ for fermions. It is a common knowledge that bosons have their spin quantized in integer multiples of $\hbar$, while fermions occur with half odd integer multiples of $\hbar$ for their spin. This two-way connection between spin and statistics is the celebrated spin–statistics theorem[2].

More generally in a d-dimensional space, the spin of the system is related to the little group, the group of transformations that leaves the momentum of the system invariant. We need, for describing them, the representation of SO(d) and its covering group spin (d). For $d \geq 3$, they come in two varieties: tensorial representations relevant for bosons and spinorial ones for fermions. In two dimension, the group being SO(2), isomorphic to $U(1)$, we have a special feature that it does not lead to quantization (of the angular momentum). Indeed, the one dimensional representations of SO(2) are labelled by θ; $0 \leq \theta < 2\pi$. Hence typically the wave function of a particle at rest at origin is given by

$$\psi(\mathbf{r}) \simeq f(r)e^{i\theta\alpha};\ \tan\alpha = y/x.$$

To describe its statistics, let us consider two identical particles at x_1 and x_2 at time t_0 and again at t_1. Quantum mechanically, the relevant amplitude gets contribution not only from the trajectories for which the particles originally at x_1 winds up at x_1 but also from those when it ends up at x_2 and its place is taken up by the particle which started from x_2. Clearly, the two classes of trajectories cannot be continuously deformed into one another. The exchange of the two particles implies propagation to an equivalent point in the configuration space of the two-particle system. Thus the process traces a closed trajectory in the configuration space. When the particles move in a plane, it is possible to trace the trajectory, using the angle the relative coordinate makes with, say x-axis, when they move about each other. For every closed trajectory, this angle α completes an integer multiple of π, of which *the odd integer* values correspond to *exchange* of two identical particles. The probability amplitude associated with the process will be given by

$$\rho(\alpha) = e^{i\theta\alpha/\pi},$$

where α is the winding angle and θ is parameter that characterizes the state. In three or more dimensions, there is no unique α, since the relevant trajectory cannot be described using a single angle, neither is the there any θ—parameter characterizing the relevant representation.

For N identical particles, the wave function is in general (when $d \geq 3$), some representation of the permutation group P_N. There are two one-dimensional representations: (i) when all elements of P_N are assigned a value 1 corresponding to bosons and (ii) even/odd elements assigned ± 1, representing fermions. For $d=2$ case, we have a much richer group B_N (Artin's braid group) of which the one-dimensional representations are characterized by a parameter θ, which interpolates between bosons ($\theta=0$) and fermions ($\theta=\pi$). For any θ, we have anyons[3]. The higher dimensional representations will have 'non abelian statistics' and it will be our endeavour to look for spin–statistics connection in this richer environment.

Topological interpretation of statistics

A classical system is specified by its configuration space Q, that has information about the positions, orientation, etc. of the constituent particles. Quantum

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mechanically, we have a related wave function ψ (q); $q \in Q$, which is a complex number. $\psi^*\psi$ is the probability density. Thus ψ represents a map from Q onto $\mathbb{C}^1$. If this is a system of N identical particles then ψ will be a one-dimensional representation of P_N (in $d \geq 3$) or B_N (in $d=2$).

We are interested in closed loops in the configuration space, since they are related to transformations linking equivalent configurations of the system. This is precisely the information, we refer to as the statistics. A loop l_1 in Q is said to be homotopic to l_2 if l_1 can be deformed continuously to l_2. All loops can be classified into different homotopic classes and they form a group. This is known as the fundamental group π_1 (Q) of the space Q. For example, if $Q=S^1$, $\pi_1(S^1)=Z$, set of integers that physically signifies the winding numbers (anticlockwise windings may be termed positive and clockwise negative). If $Q=\mathrm{SO}(3)$ then π_1 $(\mathrm{SO}(3))=Z_2$. If $\pi_1(Q)$ is nontrivial, then we expect that each *one-dimensional* representation will correspond to a different quantization. This is natural, since, for every loop, the relevant wave function obeys

$$\psi \xrightarrow{l} \mathrm{e}^{\mathrm{i}\alpha(l)}\psi,$$

and hence each representation of $\alpha(l)$ will define distinct quantum realization. Q for N identical particles is clearly

$$Q=\frac{(R^d)^N-\Delta}{S_N},$$

where $-\Delta$ ensures that no two particles are at the same site and S_N is the permutation group.

$$\pi_1(Q)=S_N \text{ for } d>2.$$

There are two one-dimensional representations of S_N. The trivial one assigns 1 to every element and this corresponds to the quantization when we treat the N particles to be bosons. When we assign 1 to every even permutation of S_N and -1 to odd permutations we have fermionic quantization. When $d=2$, we have

$$\pi_1(Q)=B_N(R^2),$$

the braid group on a plane. The group can be described as follows. Take N particles in a row along the horizontal direction, which may be taken as the x-axis of the two-dimensional plane. Let y-axis be perpendicular to the plane and vertically upwards is the time evolution. The exchange between ith and $(i+1)$th particle, σ will be represented as shown in Figure 1.

Notice that the particle in the ith position goes under the other. Clearly $\sigma_i^2 \neq 1$, as can be pictorially seen in Figure 2 through (no continuous deformation can take LHS into RHS).

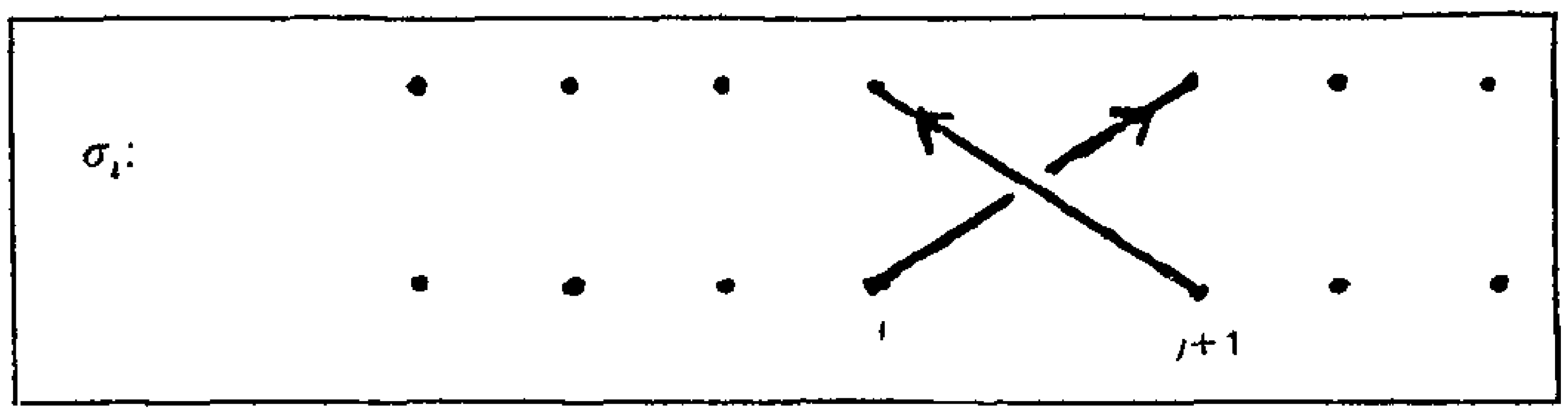

Figure 1. σ_i exchanges ith and $(i+1)$th particle.

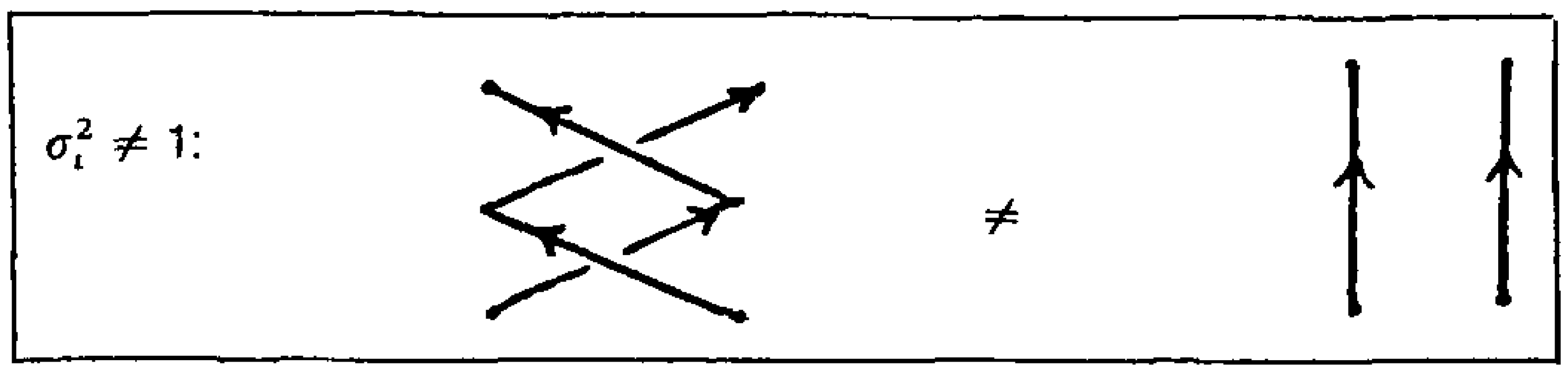

Figure 2. Illustration of $\sigma_i^2 \neq 1$.

The group is defined through

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}.$$

If the elements σ_i are to be given by a one-dimensional representation, the above relation implies that all σ_is are equal and let them be denoted by $\mathrm{e}^{i\theta}$, where θ is some parameter.

$$\psi(x_1, x_2)=\mathrm{e}^{i\theta}\,\psi(x_2, x_1).$$

While $\theta=0$ corresponds to bosons, $\theta=\pi$ to fermions, for a general θ, we have anyons. If $d>2$, necessarily (since the third and other dimensions permit the needed deformation) $\sigma_i^2=1$ and the only possibilities are $\theta=0$ and π.

Proof of spin–statistics connection

We shall now attempt to establish the spin–statistics connection, using topological arguments[4]. We will not require the full apparatus of the relativistic field theory, but only the feature that we have, along with the particle states, corresponding charge conjugates-antiparticles and the usual notion of pair creation and pair annihilation. With the time arrow vertically upward, we shall denote particles and antiparticles trajectory as shown in Figure 3.

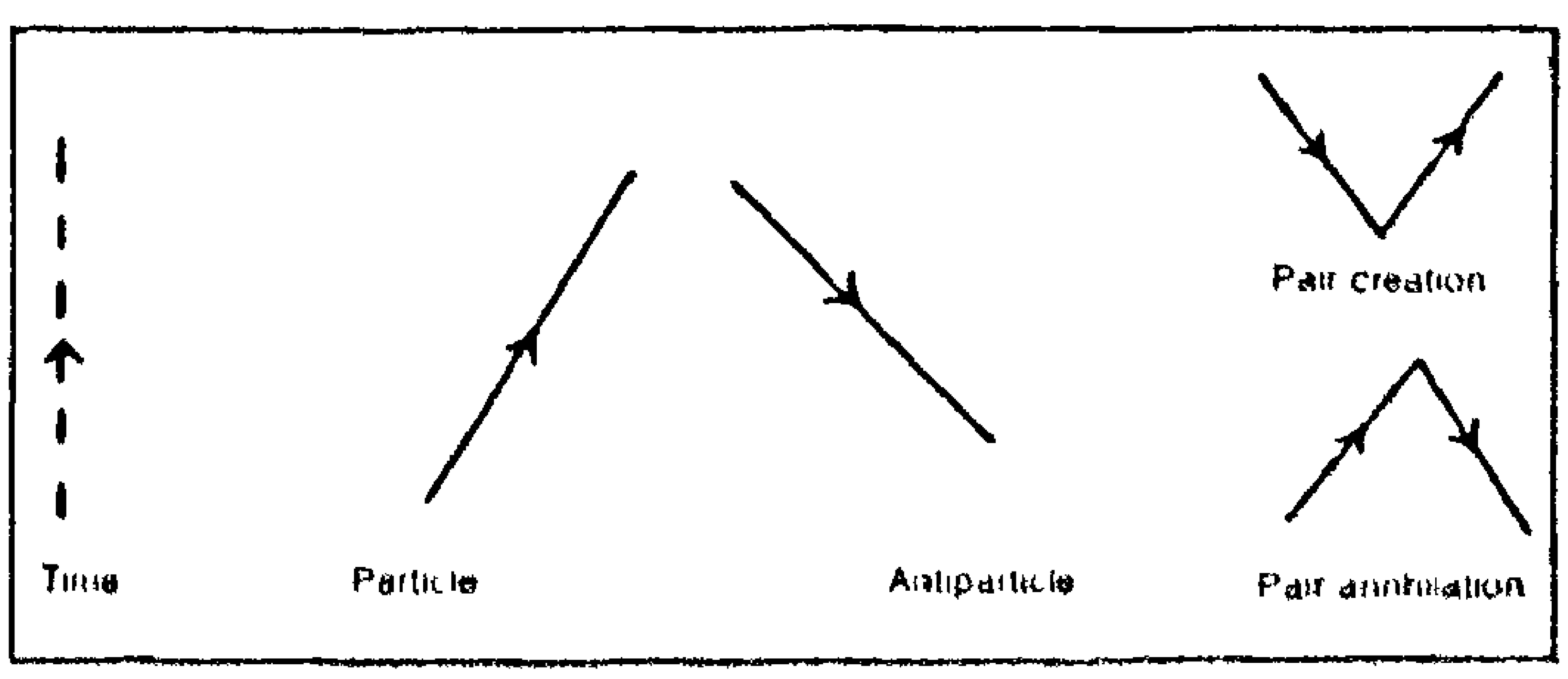

Figure 3. Particle and antiparticle trajectory.

Pair creation and annihilation occur when the particle position x_1 and antiparticle position $\bar{x}_2$ coincides, i.e. $x_1=\bar{x}_2$. For a particle with spin, it is necessary to associate with each particle a position and a frame. Thus the configuration is denoted by (x, F); position $x \in R^d$ and the orientation of a frame $F \in SO(d)$, an element of SO(d). When the particle traces a curve in R^d, the related antiparticles are described by $(\bar{x}, \bar{F}) \in R^d \odot SO(d)$ the frames $\bar{F}$ having opposite orientation with respect to particles. We denote particles with an upward-directed arrow and left-side shading and antiparticles through a downward-directed arrow and right-side shading (Figure 4). Such a convention of shading is necessary to realize pair creation and annhilation of spinning particles.

Now we are ready to give a pictorial proof of spin–statistics connection. Let us denote through σ an exchange of two identical particles, as depicted in Figure 6, which can be successively deformed to obtain the sequence shown in Figure 7.

We thus notice that the exchange of the two particles is homotopic to having one of the two particles effected a 2π rotation. Naturally the statistics, depicted by the exchange is intimately related to the action of the rotation, which characterizes its spin. This is valid for any number of spatial dimensions $d \geq 2$ and is easily generalized to arbitrary number of identical particles. Consequently, for $d>2$, symmetric (antisymmetric) states or equivalently bosons (fermions), correspond to tensorial (spinorial) representations of angular momentum. For $d=2$, $e^{i\theta}$ statistics goes with the spin (SO(2) representations) characterized by $e^{i\theta}$.

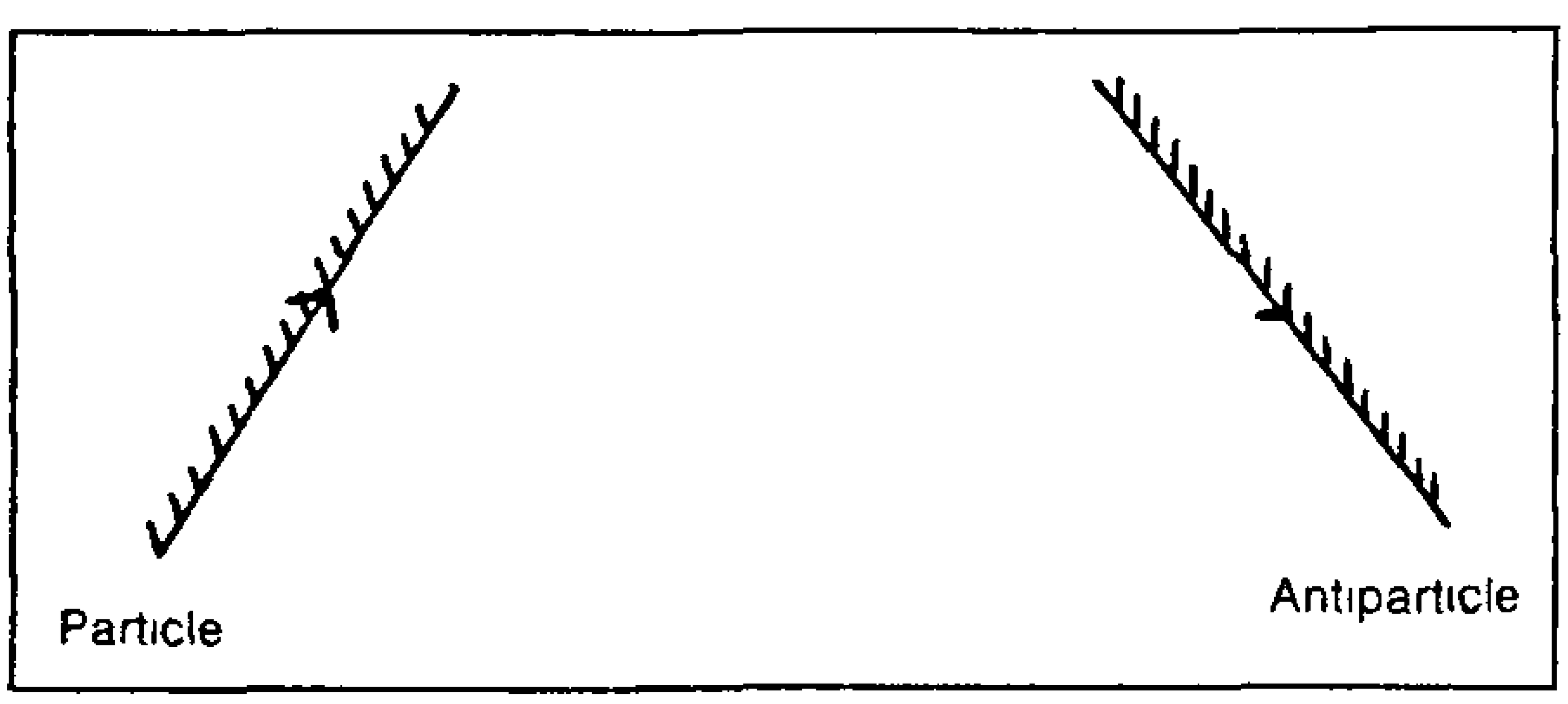

Figure 4. Notation for spinning particle, shading on one side depicts the frame carried by the particle with spin.

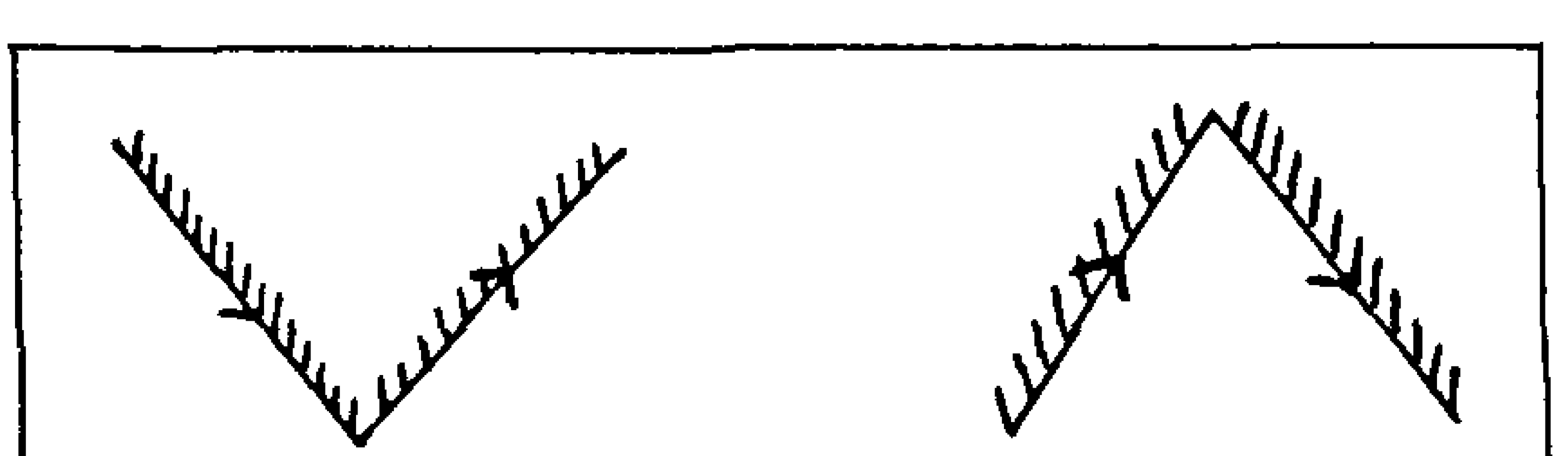

Figure 5. Pair creation and annihilation of particles with spin.

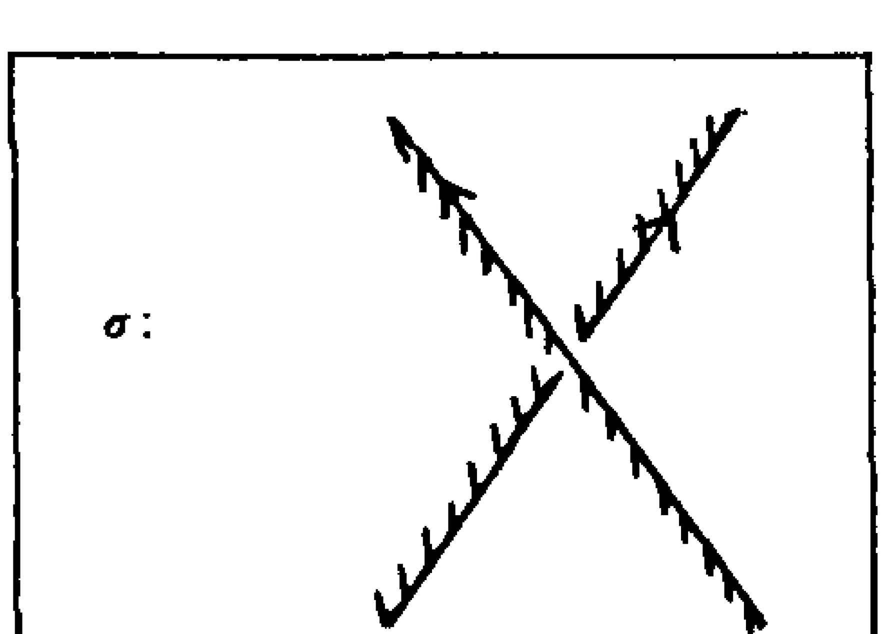

Figure 6. Exchange of two identical particles with spin.

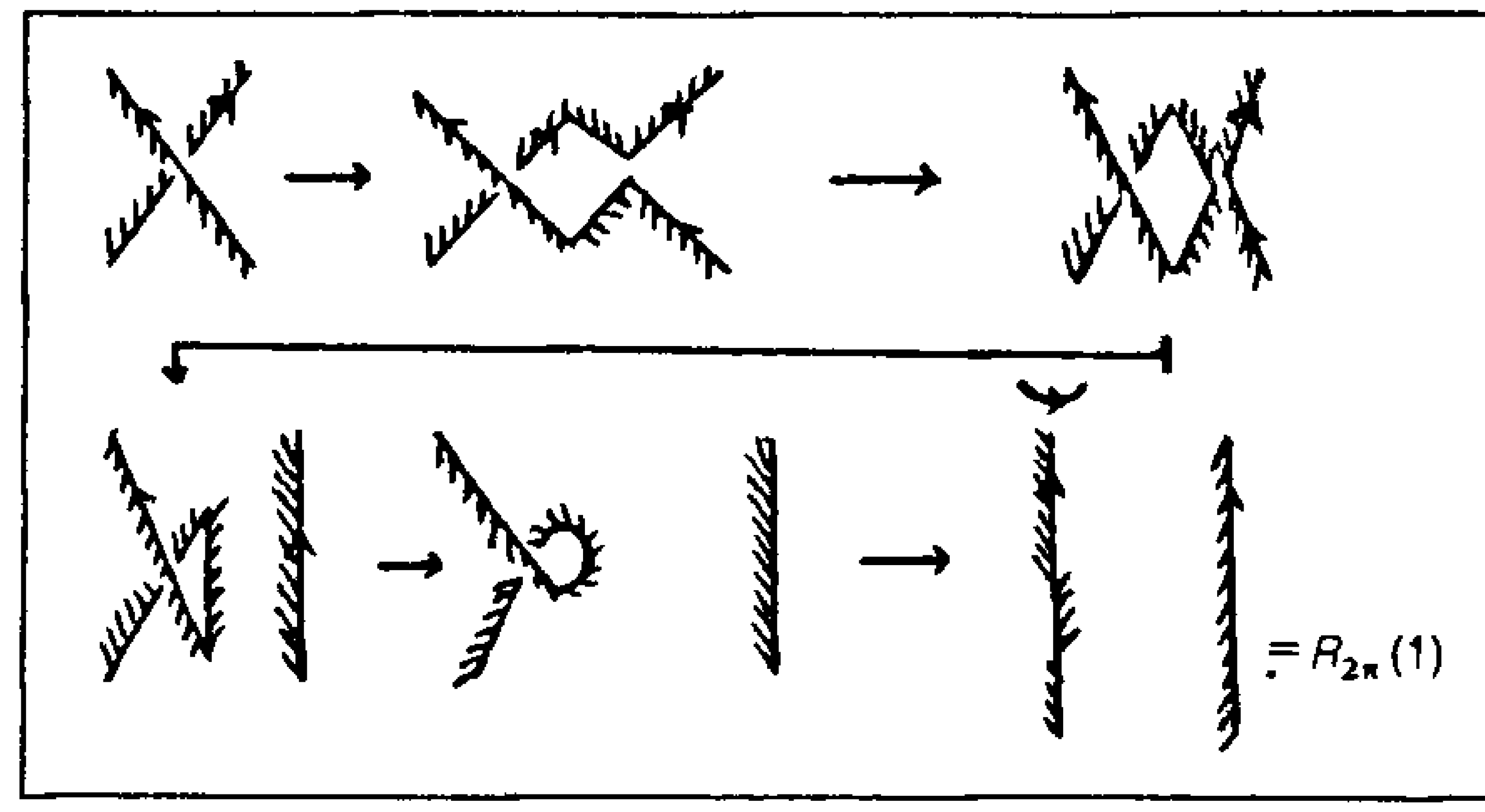

Figure 7. Pictorial proof that exchange of two identical particles is homotopic to one in which the frame of one of the two particles rotates by 2π.

Spin and statistics on compact two-dimensional surfaces

It is interesting to consider particles/systems that can be allowed on a compact two-dimensional manifolds. Let us start with a *sphere*. The relevant representations for N identical particles is that of $B_N(S^2)$, the braid group for N particles on a sphere. The allowed one-dimensional representations are given by $e^{i\theta}$ but with $\theta=(2\pi n/N-1)$; $n<N-1$. The constraints on θ are a consequence of the fact that when one of the particles is taken along a closed trajectory around the remaining $N-1$ particles it picks up a phase $e^{i(N-1)\theta}$ and this is necessarily a trivial closed trajectory (since the loop can be collapsed to a point), with the result its phase is $e^{2i n\pi}$. Except $\theta=0$, all other cases are N-dependent and therefore cannot be true as the statistics of the species. Thus only bosons are permitted on a sphere. The same conclusion follows, when we add antiparticles, in each particle number sector. This is unexpected, since we know that explicit field theory models in 2+1 dimensions admit solitons with fractional spin and statistics[5]. The puzzle is resolved when we recognize that such solutions are arrived at as finite-energy configurations on, say, planes; and it is this fact that identifies the point at infinity, leading to a closed manifold on which the field theory is described. Strictly speaking, therefore, the closed manifold of our problems comes endowed with a marked point, even though the location of the marked point is arbitrary, usually due to some global symmetry. For such base point-preserving maps, there is no constraint on θ.

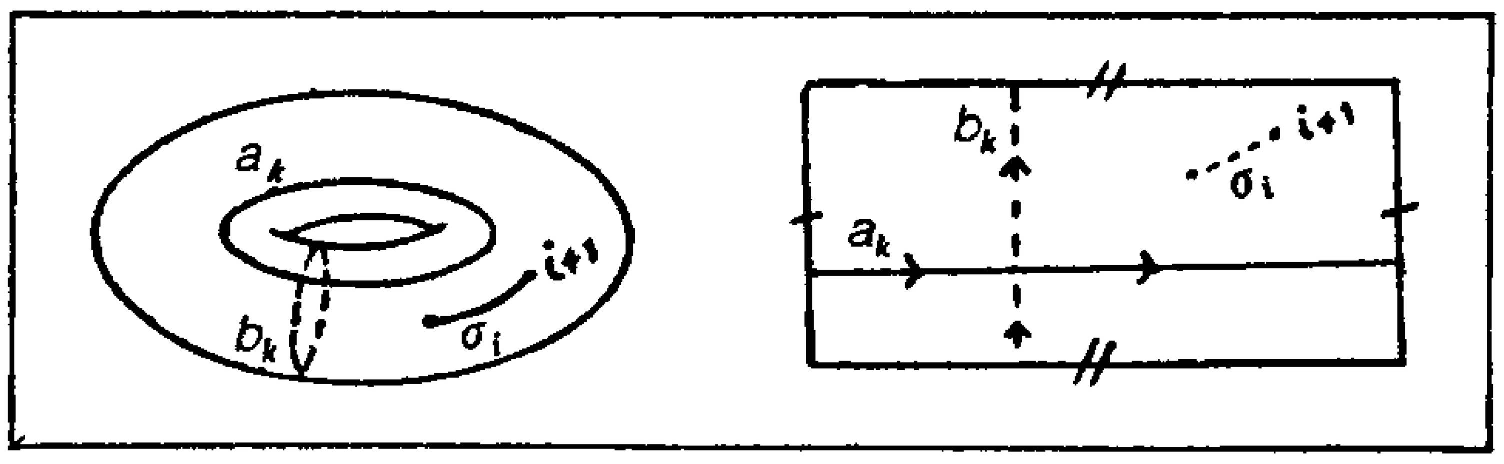

Figure 8. A torus is represented by a rectangle in which opposite sides are identified. Identical particles are placed in a row along the diagonal. σ_i exchanges ith and $(i+1)$th particle. a_k, b_k are loops that take kth particle around the two non-trivial cycles of the torus.

On a torus, we need similarly to consider $B_N(T^2)$. The closed loops on a torus are generated by the action of σ_i $(i=1,2,\ldots,N-1)$ which exchanges the ith and $(i+1)$th particles, and by a_k and b_k $(k=1,2,\ldots,N)$, the generators of the kth particle around the two nontrivial homological cycles on a torus (Figure 8).

It can be argued that the N independent one-dimensional representations correspond to $\theta=0$ and π; bosons and fermions. Fractional statistics is possible only when we consider higher dimensional representations with marked point preserving mappings.

The presentation of the group $B_N\,(T^2)$ includes apart from the usual relations,

$$\sigma_k\sigma_{k+1}\sigma_k=\sigma_{k+1}\sigma_k\sigma_{k+1}$$

and

$$\sigma_k\;\sigma_l=\sigma_l\;\sigma_k;|l-k|\geq 2.$$

relations involving a_k and b_k:

$$a_1\sigma_j=\sigma_j a_1;\; j\geq 2$$

and

$$\sigma_i^2=a_i^{-1}\;b_{i+1}\;a_i b_{i+1}^{-1}.$$

The spin–statistics connection implies that all exchanges can be related to a 2π-rotation, which we will denote as Z.

$$\sigma_i=\;\sigma_j=\;R_{2\pi}=\bar{R}_{2\pi}\equiv Z.$$

This commutes with every other generator.

It is enough to consider the generators a_1 and b_2, since the rest are reducible by the process of successive exchanges. Denote them as $a_1=X^{-1}$ and $b_2=Y$. The relevant group is characterized by the relations

$$XY=Z^2YX;\;[Z,X]=0=[Z,Y].$$

We are to find the representations of the group with these relations for the generators X, Y and Z.

Solutions

For a one-dimensional representation, obviously $Z^2=1$ and hence $Z=+1$ (bosons) and $Z=-1$ (fermions) are the only two possibilities.

For a two-dimensional representation, we may assign $X=\tau_1$, $Y=i\tau_3$ familiar Pauli matrices, leading to $Z=\pm iI$. Alternatively, it is possible to have $X=\tau_1$, $Y=i\tau_3$, so that, $Z=\pm I$. While the latter implies under exchange $\psi(q)\to\pm\psi(q)$, bosons and fermions, the former implies $\psi(q)\to\pm i\psi(q)$, the so called semions, a state midway between boson and fermion $(\theta=\pi/2)$. Generating similarly the n dimensional representation, we have

$$X=\begin{bmatrix}0 & 1 & 0 & 0 & . & . & 0\\ 0 & 0 & 1 & 0 & . & . & 0\\ . & . & . & . & . & . & 0\\ . & . & . & . & . & . & 1\\ 1 & 0 & . & . & . & . & 0\end{bmatrix}\quad Y=\begin{bmatrix}\omega & & \\ & \omega^2 & \\ & & \omega^n\end{bmatrix}$$

$$Z^2=\omega I,\;\omega=\exp(2\pi ik/n)\quad k=1,2,\ldots n-1.$$

The exchange operator $Z=\gamma\sqrt{\omega}\,I=\exp(i\pi k/n)I$ depends on n, the dimension of the representation and not on the number N of the particles. Hence, it is a valid assignment of statistics. We have in them a new species, a truly non-abelian statistics possible only in two dimensions. Perhaps there will be some interesting applications of such possibilities of fractional non-abelian statistics.

The analysis can be generalized to compact surfaces of arbitrary genus. The resultant group will have generators Z and X_i, Y_i $(i=1,2,\ldots,g)$ and representations similar to the case of a torus can be identified.

Conclusion

We have shown that the spin–statistics connection has a topological basis and needs only the notion of an antiparticle. This is extendable to two-dimensional spaces in which the fractional spin is linked to the fractional statistics. On compact surfaces, further constraints exist that cause only bosons to be sustainable on spheres; similarly only bosons and fermions are permitted on a torus. We observe that, by considering base point-preserving mappings on a torus and, similarly, higher-genus surfaces, we have states with multicomponent representations satisfying fractional statistics. These novel states may have interesting applications.

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