be a fifteen-digit number such that $10^3|N$, $10^4\nmid N$. But N is generated by $8(9)_{10}6880$.

Also, $(8)_5(9)_6000$, $40(9)_{10}000$ are generated by $(8)_5(9)_5 8890$ and $40(9)_9 8890$ respectively.

The above examples show that theorem (1) is not true for numbers $>10^{13}$ though they satisfy all other given conditions. In other words, I have indirectly shown that this theorem cannot be extended further without imposing extra conditions.

Proof of theorem (2)

$$\text{Let } N=\sum_{i=4}^{12} a_i\cdot 10^i, \quad (2.1)$$

where $0<a_4\leqslant 9$, $0\leqslant a_i\leqslant 9$, for $i=5$ to 12

($\therefore$ $d(N)\leqslant 74$) and $d(N)\equiv 8(\text{mod } 11)$.

If possible, let it be generated by

$$M=\sum_{i=0}^{12} b_i\cdot 10^i, \quad (2.2)$$

where $0\leqslant b_i\leqslant 9$ and $b_i\neq 0$ for at least one $i=0$ to 12.

$$\therefore \quad N=M+d(M)$$

$$=\sum_{i=0}^{12} b_i(10^i+1). \quad (2.3)$$

Since $10^4|N$, i.e. $N\equiv 0$ (mod 10^4),

$$\sum_{i=0}^{12} b_i+1000b_3+100b_2+10b_1+b_0\equiv 0(\text{mod } 10^4).$$

$$\therefore \quad \sum_{i=0}^{12} b_i+1000b_3+100b_2+10b_1+b_0=10^4. \quad (2.4)$$

Substituting (2.4) in (2.3), we get

$$N=b_{12}\cdot 10^{12}+b_{11}\cdot 10^{11}+\ldots+(b_4+1)10^4, \quad (2.5)$$

where $b_4+1\neq 0$ for $10^4\nmid N$.

Hence, from (2.1) and (2.5),

$$a_i=b_i \quad \text{for } 5\leqslant i\leqslant 12 \quad (2.6)$$

and $a_4=b_4+1$.

Again, from (2.4) and (2.6) we get,

$$d(N)+1001b_3+101b_2+11b_1+2b_0=10001. \quad (2.7)$$

$$\therefore \quad b_0+b_2\equiv 8(\text{mod } 11).$$

Since $0\leqslant b_0+b_2\leqslant 18$, we must have $b_0+b_2=8$

$$\Rightarrow b_0\leqslant 8 \quad \text{and} \quad b_2\leqslant 8.$$

From (2.7),

$$10001=d(N)+1001b_3+99b_2+11b_1+2(b_0+b_2)$$
$$\leqslant 74+9009+(99\times 8)+99+16=9980,$$

which is false.

Therefore, $b_0+b_2=8$ is also not possible, which shows that the solution of (2.1) for equation (2.3) does not exist.

This completes the proof of theorem (2).

Counterexamples outside the range $0\leqslant N\leqslant 10^{13}$

For a number N, $d(N)\equiv 8(\text{mod } 11)$ such that $10^4|N$, $10^5\nmid N$, and if $d(N)$ is at most 74, only then is it a self-number. In this case, since $d(N)=74$, N can be placed in at most 13 digits. Hence the range of N is the theorem is $0<N<10^{13}$. Beyond this range, though the number satisfies all other conditions, it may not be a self-number. To show this I give the following counter-examples. Here $(a)_k$ means a repeated k times in a row.

Example 1. Let $N=(9)_8 850000$, $d(N)=85\equiv 8(\text{mod } 11)$ be such that $10^4|N$, $10^5\nmid N$. But N is generated by $(9)_8 849890$.

Example 2. Let $N=(9)_{10}60000$, $d(N)=96\equiv 8(\text{mod } 11)$ be such that $10^4|N$, $10^5\nmid N$. But N is generated by $(9)_{10}59880$.

Example 3. Let $N=6(9)_8 52(0)_4$, $d(N)=85\equiv 8(\text{mod } 11)$ be such that $10^4|N$, $10^5\nmid N$. But it is generated by $M=6(9)_8 519890$.

Thus the above examples show that theorem (2) is not true for numbers $>10^{13}$, though they satisfy all the other given conditions.

In other words, I have shown that this is the best possible range.

1. Kaprekar, D. R., *The Mathematics of the New Self-Numbers*, Devlali, 1963, p. 19.
2. Vaidya, A. M. *Math. Stud.*, 1969, **37**, 212.
3. Joshi, V. S., *Math. Stud.*, 1971, **39**, 327.

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Frustrated limit cycle and irregular behaviour in a nonlinear pendulum

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We discuss how the presence of frustration brings about irregular behaviour in a pendulum with nonlinear dissipation. Here frustration arises owing to the particular choice of the dissipation. A preliminary numerical analysis is presented which indicates the transition to chaos at low frequencies of the driving force.

FRUSTRATION is a phenomenon encountered in systems with two competing interactions[1]. In many physical systems such as magnetic systems[2], amorphous packing, random networks and neural systems, frustration leads to interesting and novel consequences[3]. In this paper we introduce a system in which the presence of frustration precedes the transition to chaotic behaviour.

The system we consider is a nonlinear pendulum driven by a sinusoidal force and subjected to a damping that depends both on the velocity and the coordinates. The onset of chaotic behaviour in such a system has been studied recently[4] using Melnikov analysis as well as numerical methods. However, it appears that the route to chaos in such a system is not clearly understood. In this communication we discuss how nonlinear dissipation is of crucial significance in the origin of frustration and irregular behaviour in this system.

An ordinary pendulum, with the usual type of dissipation in which the dissipative term depends linearly on velocity, has been studied extensively[5]. Such a system is described by an equation of motion,

$$\ddot{x} = -\sin x - g\dot{x} + A\sin \omega t. \quad (1)$$

This is found to undergo a cascade of period-doubling bifurcations, which is generic, occurring in both the oscillating and rotating regimes. Chaotic behaviour also arises as a result of random transitions between two phase-locked states that have become unstable. In this system, it is the interplay between the driving force and damping term that leads to limit cycles as well as phase-locked states, which then undergoes period doubling.

The limit-cycle behaviour is considerably altered when we consider a system that is described by the equation

$$\ddot{x} = -\sin x - g\dot{x}(x^2 - 1) + A\sin \omega t. \quad (2)$$

It is clear from (2) that the nature of the velocity-dependent term is decided by whether $|x| > 1$ or $|x| < 1$, x changing sign as it crosses the value of unity. This change of sign causes qualitatively different asymptotic behaviour. As the system evolves in time, whenever $|x|$ increases beyond 1, the dissipation due to the second term brings the trajectory inwards, decreasing the value of x below 1. Then, instead of dissipation, we have a growing solution taking the system to values of x above 1. For low values of the driving amplitude and frequency, the system therefore does not settle down to any limit cycle asymptotically but goes over a set of trajectories resulting in a band-like limit cycle, which we call 'frustrated limit cycle'. In Figure 1, we illustrate this for values of $\omega = 0.4$, $A = 0.2$ and $g = 0.2$.

The power spectrum, obtained using the fast Fourier transform (FFT) corresponding to these values of ω, A and g, shows four dominant but broad peaks (Figure 2). This is in contrast to the FFT for quasiperiodic motion, which gives rise to sharp peaks. The broadening of the peaks arises as a result of the band-like nature of the limit cycle. As A is increased additional peaks appear in the FFT, indicating the presence of more frequencies, and at $A = 0.26$ we have a chaotic power spectrum (Figure 3). We calculated the maximum Lyapunov exponent λ_{max}. The variation of λ_{max} with A is given in Table 1. We find that λ_{max} becomes positive at $A = 0.21$.

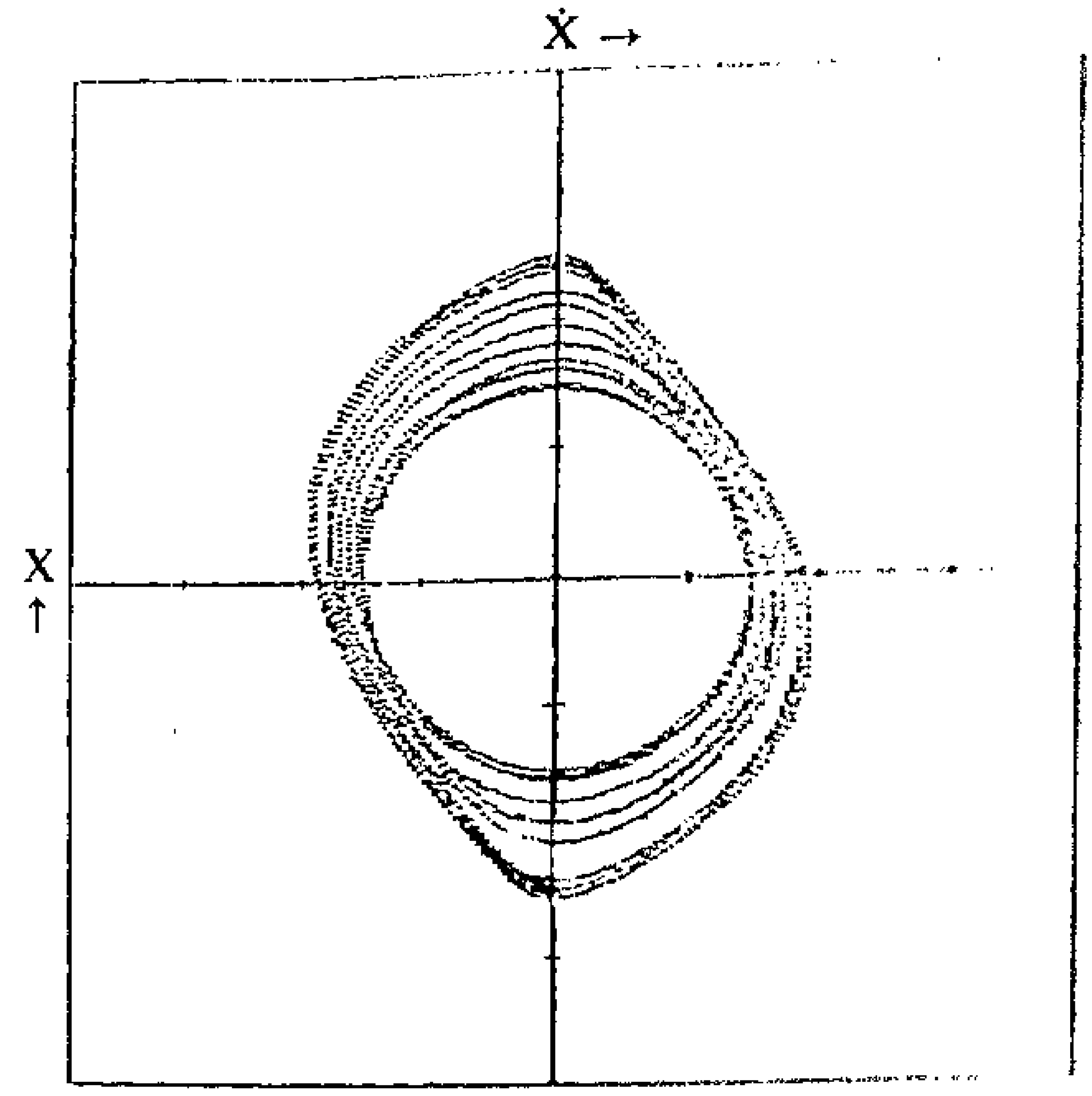

Figure 1. The frustrated limit cycle of the nonlinear pendulum for $\omega = 0.4$, $g = 0.2$ and $A = 0.2$.

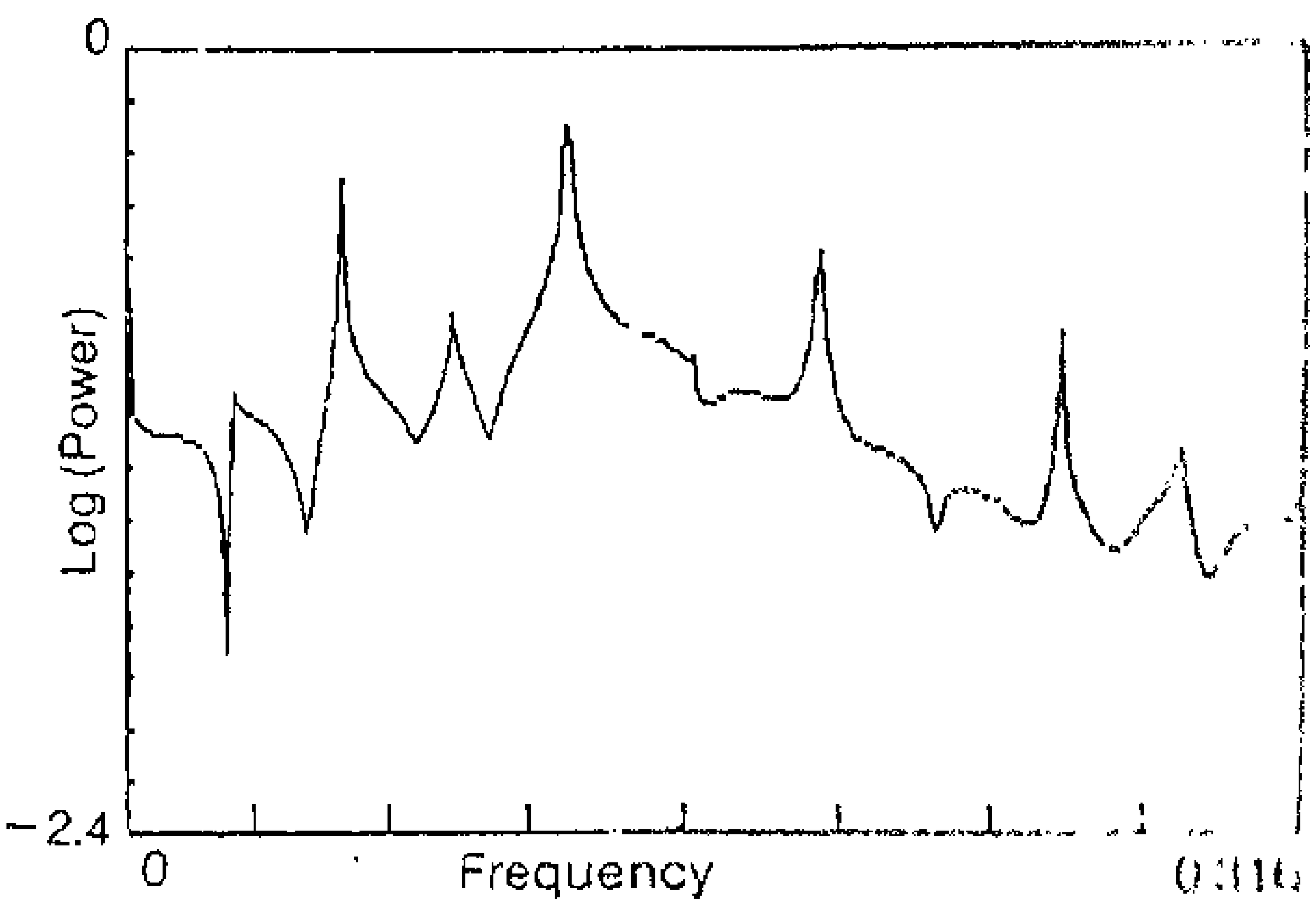

Figure 2. Power spectrum using FFT corresponding to $\omega = 0.4$, $g = 0.2$ and $A = 0.2$.

None of the general routes to chaos with which we are familiar seems to describe this transition. Increasing the value of A increases the frustration in the system, as indicated by the presence of additional frequencies in FFT and positive Lyapunov exponent. This can finally lead to chaotic behaviour before the trajectory escapes from the first potentional well. This type of behaviour is found to occur in the frequency range $0.08 < \omega < 1$. When $\omega > 1$ the frustrated limit cycle exists for small values of A. As A is increased, the band splits up into periodic cycles. However, this is not followed in any sequence and only isolated periodic bands are observed. For large enough values of A, the system asymptotically settles down to a limit cycle with the periodicity of the applied force.

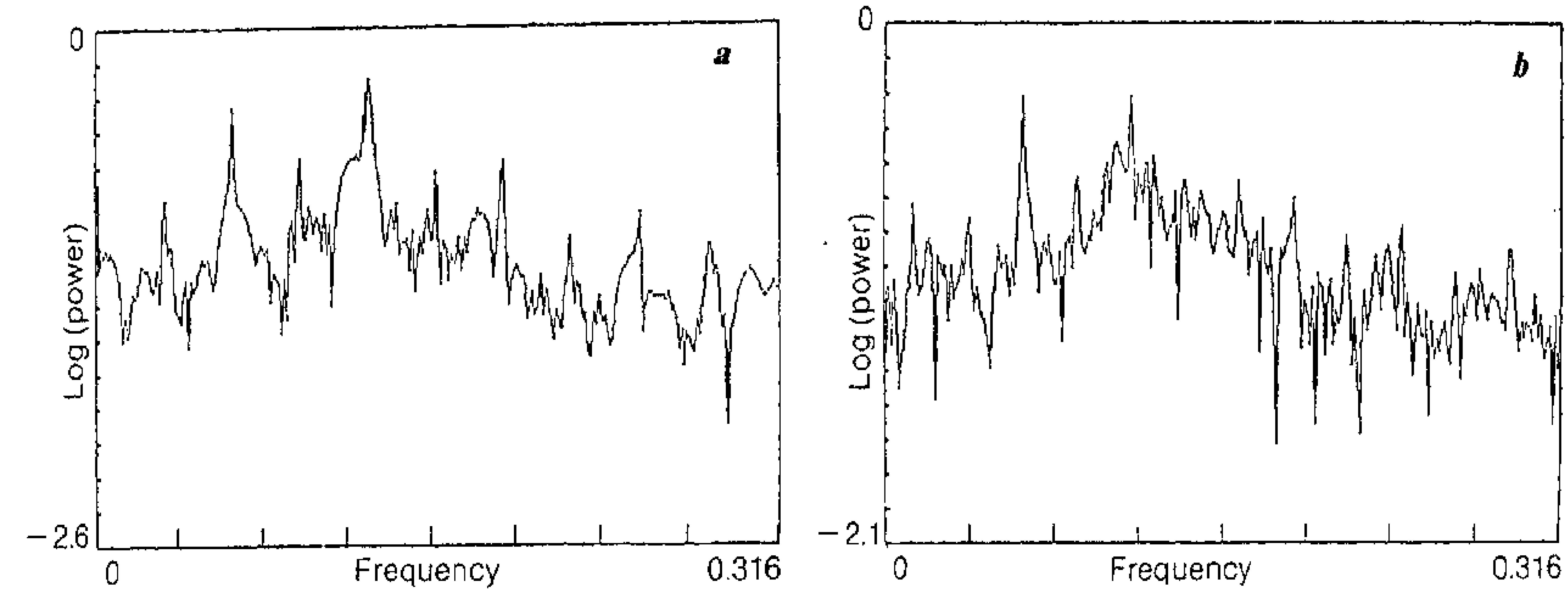

Figure 3. Power spectra showing that as A is increased additional frequencies appear in the system. In *b* ($A = 0.26$), the system shows chaotic behaviour.

We feel that the frustration makes the system extremely sensitive to changes in the external parameters, so that a small change in the control parameter can drive the system to the chaotic state. More detailed investigations are necessary before one can make definite predictions regarding the nature of the transition to chaos. We are currently investigating this and the results will be presented elsewhere.

Table 1. The maximum Lyapunov exponent λ_{max} for the system when $\omega = 0.4$, $g = 0.2$, and A is varied.

A	λ_{max}
0.15	-2.338065×10^{-3}
0.2	-6.341918×10^{-4}
0.21	5.757624×10^{-4}
0.22	1.253144×10^{-3}
0.24	1.979667×10^{-3}
0.26	3.467343×10^{-3}
0.28	3.55532×10^{-3}
0.3	4.432041×10^{-3}

1. Toulouse, G., *Commun. Phys.*, 1977, **2**, 115.
2. Toulouse, G., in *Springer Lecture Notes in Physics*, Springer, Berlin, 1983, p. 192.
3. Anderson, P. W., in *Ill Condensed Matter*, (eds. Balian, R., Meynard, R. and Toulouse, G.), North-Holland, New York, 1983.
4. Ambika, G. and Babu Joseph, K., *Pramana—J. Phys.*, 1988, **31**, 1.
5. D'Humieres, D., Beasley, M. R., Huberman, B. A. and Libhaber, A., *Phys.Rev.*, 1982 **A26**, 3483.
6. Moon, F. C., *Chaotic Vibrations*, Wiley, New York, 1987.

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Decline of ^{210}Pb fallout on Greenland in the last century

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We have carried out systematic measurements of ^{210}Pb fallout over a period of about 100 years at Dye-3 station in Greenland using a precisely dated 77-m deep ice core. The core was dated using data on annual cyclic variations in δ^{18}O, artificial radioactivity and elevated levels of acidity due to major volcanic eruptions. The results indicate that the fallout of ^{210}Pb has not remained constant over the last century and was higher by a factor of about two during 1885–1920 than in 1920–1975. Possible causes for the changes in fallout due to volcanic eruptions and nuclear explosions are discussed. If the observed trend is valid on a global scale, it raises serious doubts about the basic assumption of ^{210}Pb geochronology.

PAST records of climatic changes, atmospheric and nuclear fallouts, volcanic debris and a wealth of other information are preserved systematically in polar glaciers and ice sheets. Favourable areas for studying such deposition events are the high-latitude regions of large ice sheets, such as Greenland in the northern hemisphere, which is fed by relatively frequent and heavy snowfalls[1]. Greenland ice cores are most suitable for dating the annual layers of snow deposition using very sensitive δ^{18}O and past-acidity records[2]. The natural ^{210}Pb background in Greenland being low compared to that in other locations in the northern hemisphere because of its remoteness from natural sources, it is easy to observe even small changes in the deposition flux of ^{210}Pb caused by natural or artificial events, such as volcanic eruptions or thermonuclear explosions. Analytical study of a well-dated core from