Validity of tests of self-numbers

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Self-numbers were defined by Kaprekar as numbers M having no solutions in N for the equation $M = N + d(N)$, where $d(N)$ is the sum of digits of N, when represented in the decimal scale. In this paper I describe certain tests for numbers of the form $A \cdot 10^n$ where A and n are natural numbers, and examine to what extent the tests remain valid.

LET $d(N)$ denote the sum of the digits of N, when N is represented in the usual decimal scale. If a natural number M does not have any solution for the equation

$$M = N + d(N)$$

in natural numbers, then M is said to be a self-number. If $M = N + d(N)$ for some natural number N, then M is said to be a generated number and in this case N is called the generator of M. Equivalently, N generates M.

For example 20 is a self-number. The initial self-numbers are 1, 3, 5, 7 and 9. The number 21 is not a self-number as 15 generates 21. Some numbers may have more than one generator; the number 101 has two generators, viz. 91 and 100.

The concept of self-numbers was introduced by Kaprekar[1] more than twenty years ago. He described many conjectures about self-numbers, among which: If a number ends in 00 but does not end in 000, i.e. it is a multiple of 100 and not a multiple of 1000, and if $d(N)$ is 4, 15, 26, 37, then N is a self-number.

Vaidya[2] showed that this conjecture is not valid in general by producing counterexamples. But Vaidya also conjectured that Kaprekar's tests for self-numbers are true for all natural numbers $< 10^{11}$. Joshi[3] proved that if $10^2 | N, 10^3 \nmid N, d(N) \equiv 4 \pmod{11}$ and $0 < N < 10^{11}$, then Kaprekar's conjecture is valid.

In this paper I describe similar tests for self-numbers and show to what extent they remain valid. Actually, I prove the following:

THEOREM 1: *If $0 < N < 10^{13}$, $10^3 | N, 10^4 \nmid N, d(N) \equiv 6 \pmod{11}$, then N is a self-number.*

THEOREM 2: *If $0 < N < 10^{13}$, $10^4 | N$, $10^5 \nmid N$, $d(N) \equiv 8 \pmod{11}$, then N is a self-number.*

I describe the proofs of theorems (1) and (2) and show that the range of validity for these tests of self-numbers cannot be extended without imposing an extra condition. This I do by producing suitable counterexamples.

The method is elementary and the results are obtained by considering simple congruence relations modulo 11 and deducing contradictions from them.

Proof of theorem (1)

Let $$N = \sum_{i=3}^{12} a_i \cdot 10^i, \tag{1.1}$$

where $0 \leqslant a_i \leqslant 9$ for $i = 3$ to 12, $a_3 \neq 0$, ($\therefore$ $d(N) \leqslant 83$), and $d(N) \equiv 6 \pmod{11}$.

If possible, let it be generated by

$$M = \sum_{i=0}^{12} b_i \cdot 10^i, \tag{1.2}$$

where $0 \leqslant b_i \leqslant 9$ and $b_i \neq 0$ for at least one $i = 0$ to 12.

$$\therefore \quad N = M + d(M)$$
$$= \sum_{i=0}^{12} b_i(10^i + 1). \tag{1.3}$$

Since $10^3 | N$, i.e. $N \equiv 0 \pmod{10^3}$,

$$\sum_{i=0}^{12} b_i + 100 b_2 + 10 b_1 + b_0 \equiv 0 \pmod{10^3}$$

$$\therefore \quad \sum_{i=0}^{12} b_i + 100 b_2 + 10 b_1 + b_0 = 10^3. \tag{1.4}$$

Substituting (1.4) in (1.3), we get

$$N = b_{12} \cdot 10^{12} + b_{11} \cdot 10^{11} + \ldots + (b_3 + 1) 10^3, \tag{1.5}$$

where $b_3 + 1 \neq 10$ for $10^4 \nmid N$.

Hence, from (1.1) and (1.5),

$$a_i = b_i \quad \text{for } 4 \leqslant i \leqslant 12 \tag{1.6}$$

and $a_3 = b_3 + 1$.

Using (1.6) in (1.4), we get

$$d(N) + 101 b_2 + 11 b_1 + 2 b_0 = 1001. \tag{1.7}$$

Taking congruence modulo 11 on both sides, we get

$$b_0 + b_2 \equiv 8 \pmod{11}.$$

But $0 \leqslant b_0 + b_2 \leqslant 18$.

$$\therefore \quad b_0 + b_2 = 8$$
$$\Rightarrow b_2 \leqslant 8 \quad (\text{and } b_0 \leqslant 8).$$

From (1.7),

$$1001 = d(N) + 99 b_2 + 11 b_1 + 2(b_0 + b_2)$$
$$\leqslant 83 + (99 \times 8) + 99 + 16 = 990, \text{ which is false,}$$

showing that $b_0 + b_2 = 8$ is not possible.

Therefore, solution of (1.1) for equation (1.3) does not exist. Thus, theorem (1) follows.

Counterexamples outside the range $0 < N < 10^{13}$

For a number N, $d(N) \equiv 6 \pmod{11}$ such that $10^3 | N$, $10^4 \nmid N$, and if $d(N)$ is at most 83, only then is it a self-number. Thus, if $d(N) = 83$, then such number N can be placed in at most 13 digits. Hence the range of N in the theorem is $0 < N < 10^{13}$. Beyond this range the theorem does not hold good. To show this I give the following counterexamples. Here $(a)_k$ means a repeated k times in a row.

Example 1. Let $N = (9)_{10}4000$, $d(N) = 94 \equiv 6 \pmod{11}$ be a fourteen-digit number such that $10^3 | N$, $10^4 \nmid N$. But N is not a self-number as it is generated by $M = (9)_{10}3890$.

Example 2. Let $N = 8(9)_{10}7000$, $d(N) = 105 \equiv 6 \pmod{11}$

be a fifteen-digit number such that $10^3|N$, $10^4 \nmid N$. But N is generated by $8(9)_{10}6880$.

Also, $(8)_5(9)_6000$, $40(9)_{10}000$ are generated by $(8)_5(9)_5 8890$ and $40(9)_9 8890$ respectively.

The above examples show that theorem (1) is not true for numbers $> 10^{13}$ though they satisfy all other given conditions. In other words, I have indirectly shown that this theorem cannot be extended further without imposing extra conditions.

Proof of theorem (2)

$$\text{Let } N = \sum_{i=4}^{12} a_i \cdot 10^i, \tag{2.1}$$

where $0 < a_4 \leqslant 9$, $0 \leqslant a_i \leqslant 9$, for $i = 5$ to 12

($\therefore$ $d(N) \leqslant 74$) and $d(N) \equiv 8 \pmod{11}$.

If possible, let it be generated by

$$M = \sum_{i=0}^{12} b_i \cdot 10^i, \tag{2.2}$$

where $0 \leqslant b_i \leqslant 9$ and $b_i \neq 0$ for at least one $i = 0$ to 12.

$$\therefore \quad N = M + d(M)$$

$$= \sum_{i=0}^{12} b_i(10^i + 1). \tag{2.3}$$

Since $10^4 | N$, i.e. $N \equiv 0 \pmod{10^4}$,

$$\sum_{i=0}^{12} b_i + 1000b_3 + 100b_2 + 10b_1 + b_0 \equiv 0 \pmod{10^4}.$$

$$\therefore \quad \sum_{i=0}^{12} b_i + 1000b_3 + 100b_2 + 10b_1 + b_0 = 10^4. \tag{2.4}$$

Substituting (2.4) in (2.3), we get

$$N = b_{12} \cdot 10^{12} + b_{11} \cdot 10^{11} + \ldots + (b_4 + 1)10^4, \tag{2.5}$$

where $b_4 + 1 \neq 0$ for $10^4 \nmid N$.

Hence, from (2.1) and (2.5),

$$a_i = b_i \quad \text{for } 5 \leqslant i \leqslant 12 \tag{2.6}$$

and $a_4 = b_4 + 1$.

Again, from (2.4) and (2.6) we get,

$$d(N) + 1001b_3 + 101b_2 + 11b_1 + 2b_0 = 10001. \tag{2.7}$$

$$\therefore \quad b_0 + b_2 \equiv 8 \pmod{11}.$$

Since $0 \leqslant b_0 + b_2 \leqslant 18$, we must have $b_0 + b_2 = 8$

$$\Rightarrow b_0 \leqslant 8 \quad \text{and} \quad b_2 \leqslant 8.$$

From (2.7),

$$10001 = d(N) + 1001b_3 + 99b_2 + 11b_1 + 2(b_0 + b_2)$$
$$\leqslant 74 + 9009 + (99 \times 8) + 99 + 16 = 9980,$$

which is false.

Therefore, $b_0 + b_2 = 8$ is also not possible, which shows that the solution of (2.1) for equation (2.3) does not exist.

This completes the proof of theorem (2).

Counterexamples outside the range $0 \leqslant N \leqslant 10^{13}$

For a number N, $d(N) \equiv 8 \pmod{11}$ such that $10^4|N$, $10^5 \nmid N$, and if $d(N)$ is at most 74, only then is it a self-number. In this case, since $d(N) = 74$, N can be placed in at most 13 digits. Hence the range of N is the theorem is $0 < N < 10^{13}$. Beyond this range, though the number satisfies all other conditions, it may not be a self-number. To show this I give the following counter-examples. Here $(a)_k$ means a repeated k times in a row.

Example 1. Let $N = (9)_8 850000$, $d(N) = 85 \equiv 8 \pmod{11}$ be such that $10^4 | N$, $10^5 \nmid N$. But N is generated by $(9)_8 849890$.

Example 2. Let $N = (9)_{10} 60000$, $d(N) = 96 \equiv 8 \pmod{11}$ be such that $10^4 | N$, $10^5 \nmid N$. But N is generated by $(9)_{10} 59880$.

Example 3. Let $N = 6(9)_8 52(0)_4$, $d(N) = 85 \equiv 8 \pmod{11}$ be such that $10^4 | N$, $10^5 \nmid N$. But it is generated by $M = 6(9)_8 519890$.

Thus the above examples show that theorem (2) is not true for numbers $> 10^{13}$, though they satisfy all the other given conditions.

In other words, I have shown that this is the best possible range.

1. Kaprekar, D. R., *The Mathematics of the New Self-Numbers*, Devlali, 1963, p. 19.
2. Vaidya, A. M. *Math. Stud.*, 1969, 37, 212.
3. Joshi, V. S., *Math. Stud.*, 1971, 39, 327.

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Frustrated limit cycle and irregular behaviour in a nonlinear pendulum

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We discuss how the presence of frustration brings about irregular behaviour in a pendulum with nonlinear dissipation. Here frustration arises owing to the particular choice of the dissipation. A preliminary numerical analysis is presented which indicates the transition to chaos at low frequencies of the driving force.

FRUSTRATION is a phenomenon encountered in systems with two competing interactions[1]. In many physical systems such as magnetic systems[2], amorphous packing, random networks and neural systems, frustration leads to interesting and novel consequences[3]. In this paper we introduce a system in which the presence of frustration precedes the transition to chaotic behaviour.