Validity of tests of self-numbers

R. B. Patel
Arts, Science and Commerce College, Shahada 425 409, India

Self-numbers were defined by Kaprekar as numbers $M$ having no solutions in $N$ for the equation $M = N + d(N)$, where $d(N)$ is the sum of digits of $N$, when represented in the decimal scale. In this paper I describe certain tests for numbers of the form $A \cdot 10^n$ where $A$ and $n$ are natural numbers, and examine to what extent the tests remain valid.

Let $d(N)$ denote the sum of the digits of $N$, when $N$ is represented in the usual decimal scale. If a natural number $M$ does not have any solution for the equation $M = N + d(N)$ in natural numbers, then $M$ is said to be a self-number. If $M = N + d(N)$ for some natural number $N$, then $M$ is said to be a generated number and in this case $N$ is called the generator of $M$. Equivalently, $N$ generates $M$.

For example $20$ is a self-number. The initial self-numbers are $1, 3, 5, 7$ and $9$. The number $21$ is not a self-number as $15$ generates $21$. Some numbers may have more than one generator; the number $101$ has two generators, viz. $91$ and $100$.

The concept of self-numbers was introduced by Kaprekar$^2$ more than twenty years ago. He described many conjectures about self-numbers, among which: If a number ends in $00$ but does not end in $000$, i.e. it is a multiple of 100 and not a multiple of 1000, and if $d(N)$ is $4, 15, 26, 37$, then $N$ is a self-number.

Vaidya$^3$ showed that this conjecture is not valid in general by producing counterexamples. But Vaidya also conjectured that Kaprekar’s tests for self-numbers are true for all natural numbers $< 10^{11}$. Joshi$^4$ proved that if $10^3 | N, 10^4 | N, d(N) = 4 \pmod{11}$ and $0 < N < 10^{13}$, then Kaprekar’s conjecture is valid.

In this paper I describe similar tests for self-numbers and show to what extent they remain valid. Actually, I prove the following:

**Theorem 1:** If $0 < N < 10^{13}, 10^3 | N, 10^4 | N, d(N) = 6 \pmod{11}$, then $N$ is a self-number.

**Theorem 2:** If $0 < N < 10^{13}, 10^3 | N, 10^4 | N, d(N) = 8 \pmod{11}$, then $N$ is a self-number.

I describe the proofs of theorems (1) and (2) and show that the range of validity for these tests of self-numbers cannot be extended without imposing an extra condition. This I do by producing suitable counterexamples.

The method is elementary and the results are obtained by considering simple congruence relations modulo $11$ and deducing contradictions from them.

**Proof of theorem (1)**

Let $N = \sum_{i=3}^{12} a_i \cdot 10^i$, \hspace{1cm} (1.1)

where $0 \leq a_i \leq 9$ for $i = 3$ to 12, $a_3 \neq 0$, $d(N) \neq 83$, and $d(N) \equiv 6 \pmod{11}$.

If possible, let it be generated by

$M = \sum_{i=3}^{12} b_i \cdot 10^i$, \hspace{1cm} (1.2)

where $0 \leq b_i \leq 9$ and $b_i \neq 0$ for at least one $i = 0$ to 12.

$\therefore \hspace{1cm} N = M + d(M) \hspace{1cm} (1.3)

\begin{align*}
&M = \sum_{i=0}^{12} b_i (10^i + 1), \\
&\therefore \hspace{1cm} N = \sum_{i=0}^{12} b_i (10^i + 1).
\end{align*}

Since $10^3 | N$, i.e. $N \equiv 0 \pmod{10^3}$,

\begin{align*}
&\sum_{i=0}^{12} b_i + 100 b_3 + 10 b_2 + b_1 + b_0 \equiv 0 \pmod{10^3} \\
&\therefore \hspace{1cm} \sum_{i=0}^{12} b_i + 100 b_3 + 10 b_2 + b_1 + b_0 = 10^3.
\end{align*}

Substituting (1.4) in (1.3), we get

$N = b_{12} \cdot 10^{12} + b_{11} \cdot 10^{11} + \ldots + (b_3 + 1)10^3,$ \hspace{1cm} (1.5)

where $b_3 + 1 \neq 10$ for $10^4 \nmid N$.

Hence, from (1.1) and (1.5),

$\therefore \hspace{1cm} a_i = b_i$ for $4 \leq i \leq 12$ \hspace{1cm} (1.6)

and $a_3 = b_3 + 1$.

Using (1.6) in (1.4), we get

$N = b_{12} \cdot 10^{12} + b_{11} \cdot 10^{11} + 2b_0 = 1001.$ \hspace{1cm} (1.7)

Taking congruence modulo $11$ on both sides, we get

$b_0 + b_2 \equiv 0 \pmod{11}. \hspace{1cm} (1.8)

But

$0 \leq b_0 + b_2 \leq 18$.

$\therefore \hspace{1cm} b_0 + b_2 = 8$ \hspace{1cm} (1.9)

From (1.7),

$1001 = d(N) + 99 b_3 + 11 b_2 + 2b_0 = 1001.$

$\leq 83 + (99 \times 8) + 96 + 16 = 990$, which is false, showing that $b_3 + 1 = 8$ is not possible.

Therefore, solution of (1.1) for equation (1.3) does not exist. Thus, theorem (1) follows.

**Counterexamples outside the range $0 < N < 10^{13}$**

For a number $N$, $d(N) \equiv 6 \pmod{11}$ such that $10^3 \nmid N,$ $10^4 \nmid N$, and if $d(N)$ is at most 83, only then is it a self-number.

Thus, if $d(N) = 83$, then such number $N$ can be placed in at most 13 digits. Hence the range of $N$ in the theorem is $0 < N < 10^{13}$. Beyond this range the theorem does not hold good. To show this I give the following counterexamples. Here $(a)_k$ means a repeated $k$ times in a row.

**Example 1.** Let $N = (9)_{10}4000$, $d(N) = 94 \equiv 6 \pmod{11}$ be a fourteen-digit number such that $10^3 \nmid N,$ $10^4 \nmid N$. But $N$ is not a self-number as it is generated by $M = (9)_{10}3890$.

**Example 2.** Let $N = (8)_{10}7000$, $d(N) = 105 \equiv 6 \pmod{11}$
be a fifteen-digit number such that \(10^3 \div N, 10^4 \div N\). But \(N\) is generated by \(8(9)_8 \times 8880\).

Also, \((8)_8(9)_8 0000, 40(9)_8 0000\) are generated by \((8)_8(9)_8 8890\) and \(40(9)_8 8890\) respectively.

The above examples show that theorem (1) is not true for numbers > \(10^{13}\) though they satisfy all other given conditions. In other words, I have indirectly shown that this theorem cannot be extended further without imposing extra conditions.

**Proof of theorem (2)**

Let \(N = \sum_{i=0}^{12} a_i \times 10^i\).

where \(0 \leq a_i \leq 9\), \(0 \leq a_0 \leq 9\), for \(i = 5\) to \(12\).

\(\therefore\) \(d(N) \leq 74\) and \(d(N) \equiv 8 \pmod{11}\).

If possible, let it be generated by

\[
M = \sum_{i=0}^{12} b_i \times 10^i
\]

where \(0 \leq b_i \leq 9\) and \(b_i \neq 0\) for at least one \(i = 0\) to \(12\).

\(\therefore\) \(N = d(M)\)

\[
= \sum_{i=0}^{12} b_i (10^i + 1).
\]

Since \(10^4 \div N\), i.e. \(N \equiv 0 \pmod{10^4}\),

\[
\sum_{i=0}^{12} b_i + 10000 b_3 + 100 b_2 + b_1 + b_0 \equiv 0 \pmod{10^4}.
\]

\(\therefore\) \(N = \sum_{i=0}^{12} b_i + 10000 b_3 + 100 b_2 + b_1 + b_0 = 10^4\).

Substituting (2.4) in (2.3), we get

\[
N = b_{12} \times 10^{12} + b_{11} \times 10^{11} + \ldots + (b_4 + 1)10^4,
\]

where \(a_i = 1 \neq 0\) for \(10^4 \div N\).

Hence, from (2.1) and (2.5),

\(a_i = b_i\) for \(5 \leq i \leq 12\).

and \(a_4 = b_4 + 1\).

Again, from (2.4) and (2.6) we get,

\[
d(N) + 1001 b_3 + 101 b_2 + 11 b_1 + 2 b_0 = 10001.
\]

\(\therefore\) \(b_0 + b_2 \equiv 8 \pmod{11}\).

Since \(0 \leq b_0 + b_2 \leq 18\), we must have \(b_0 + b_2 = 8\)

which is false.

Therefore, \(b_0 + b_2 = 8\) is also not possible, which shows that the solution of (2.1) for equation (2.3) does not exist.

This completes the proof of theorem (2).

**Counterexamples outside the range \(0 \leq N \leq 10^{13}\)**

For a number \(N\), \(d(N) \equiv 8 \pmod{11}\) such that \(10^4 \div N\), \(10^5 \div N\), and if \(d(N)\) is at most 74, then only then is it a self-number. In this case, since \(d(N) = 74\), \(N\) can be placed in at most 13 digits. Hence the range of \(N\) is the theorem is \(0 < N < 10^{13}\). Beyond this range, though the number satisfies all other conditions, it may not be a self-number. To show this I give the following counter-examples. Here \(a_i\) means a repeated \(k\) times in a row.

**Example 1.** Let \(N = (9)_8 850000\), \(d(N) = 85 \equiv 8 \pmod{11}\) be such that \(10^4 \div N\), \(10^5 \div N\). But \(N\) is generated by \((9)_8 849880\).

**Example 2.** Let \(N = (9)_8 850000\), \(d(N) = 96 \equiv 8 \pmod{11}\) be such that \(10^4 \div N\), \(10^5 \div N\). But \(N\) is generated by \((9)_8 859880\).

**Example 3.** Let \(N = (9)_8 850000\), \(d(N) = 80 \equiv 8 \pmod{11}\) be such that \(10^4 \div N\), \(10^5 \div N\). But it is generated by \(M = 8(9)_8 8519890\).

Thus the above examples show that theorem (2) is not true for numbers \(> 10^{13}\), though they satisfy all the other given conditions.

In other words, I have shown that this is the best possible range.

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**Frustrated limit cycle and irregular behaviour in a nonlinear pendulum**

G. Ambika and V. M. Nandakumar

Department of Physics, Maharaja's College, Cochin 682 011, India

*Department of Physics, Cochin University of Science and Technology, Cochin 682 022, India

We discuss how the presence of frustration brings about irregular behaviour in a pendulum with nonlinear dissipation. Here frustration arises owing to the particular choice of the dissipation. A preliminary numerical analysis is presented which indicates the transition to chaos at low frequencies of the driving force.

Frustration is a phenomenon encountered in systems with two competing interactions. In many physical systems such as magnetic systems, amorphous packing, random networks and neural systems, frustration leads to interesting and novel consequences. In this paper we introduce a system in which the presence of frustration precedes the transition to chaotic behaviour.