ARROW'S PARADOX AND THE PERCENTAGE VOTING SYSTEM

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ABSTRACT

Making use of graph theory, a simple proof of Arrow's Theorem is given. To circumvent Arrow's Paradox a new voting system is proposed. It is suggested that this system called Percentage Voting System be used in our Presidential election.

INTRODUCTION

In 1951 Prof. Kenneth Arrow dealt a devastating blow to democracy when he proved that there is no such thing as an ideal voting system. As practised around the world, the inputs to any voting system are the preferences of the voters for their candidates. Arrow showed that if the input to the voting system is the preference order of each voter, then there is no reasonable way to associate a preference order for the society as a whole[1,2].

There are two types of graphs we should discuss before we investigate Arrow's Paradox. They are: (i) Strict order, and (ii) Complete order. A *strict order* is characterized by the properties: (a) irreflexivity; and (b) transitivity. An example of a strict order is shown in figure 1.

The graph satisfies irreflexivity since it has no loops at any of the nodes. A graph is transitive if, whenever (directed) edges (x,y) and (y,z) exist in the graph, (x,z) also exists. The graph in figure 1 obviously satisfies transitivity. Since both irreflexivity and transitivity are satisfied by the graph, it is a strict order.

A *complete order* is characterized by the properties: (a) completeness, and (b) transitivity. A graph is complete if there is at least one edge, in either direction, between any pair of nodes x and y (even if x and y are the same). An example of a complete order is shown in figure 2.

Note that a complete graph has to be reflexive. Also note that the complement of a strict order is always a complete order. Graph shown in figure 2 is the complement of the graph in figure 1.

The (strict) preferences of a voter for the candidates in an election can be represented by a strict order. Similarly the non-preferences of a voter between the candidates can be represented by a complete order. We take these facts as obvious. The graph in figure 1 shows preference of a voter to the candidates a, b, c and d, the first preference being for b, the second equal preferences for a and d and the third and last preference for c. Figure 2 also represents the same information. We could have used either strict order or complete order for representing the preference of a voter, but in the following analysis we will utilize both the graphs together as shown in figure 3.

We will call the resulting chromatic graph a *preference* order. The use of the chromatic graph makes the analysis of Arrow's Paradox quite simple. Before we proceed further we will state three facts about the preference order.

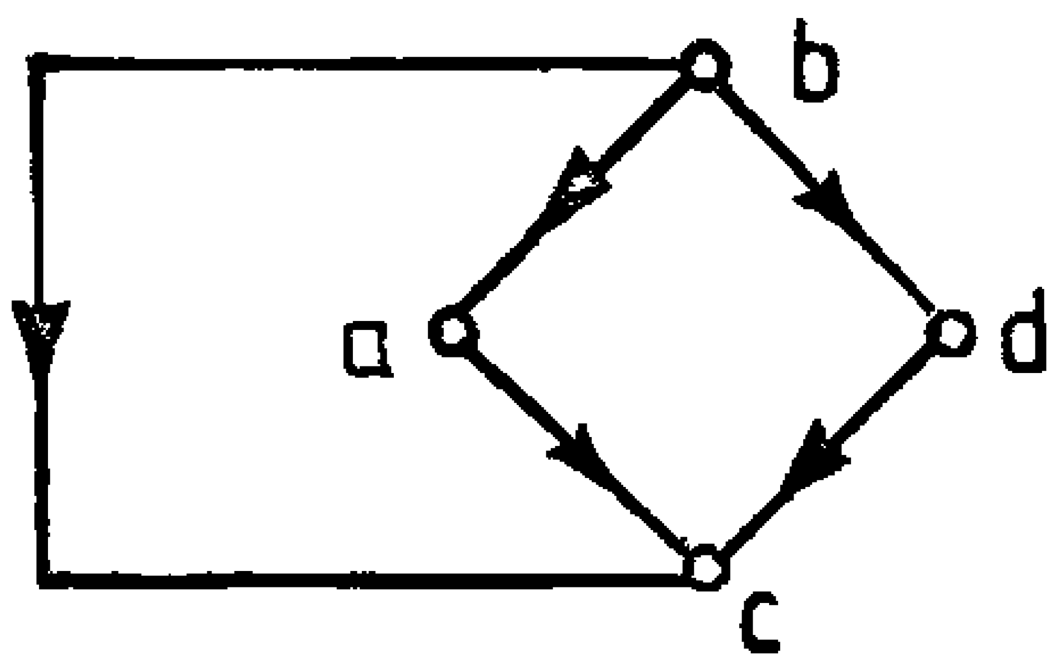


Figure 1. Strict order.

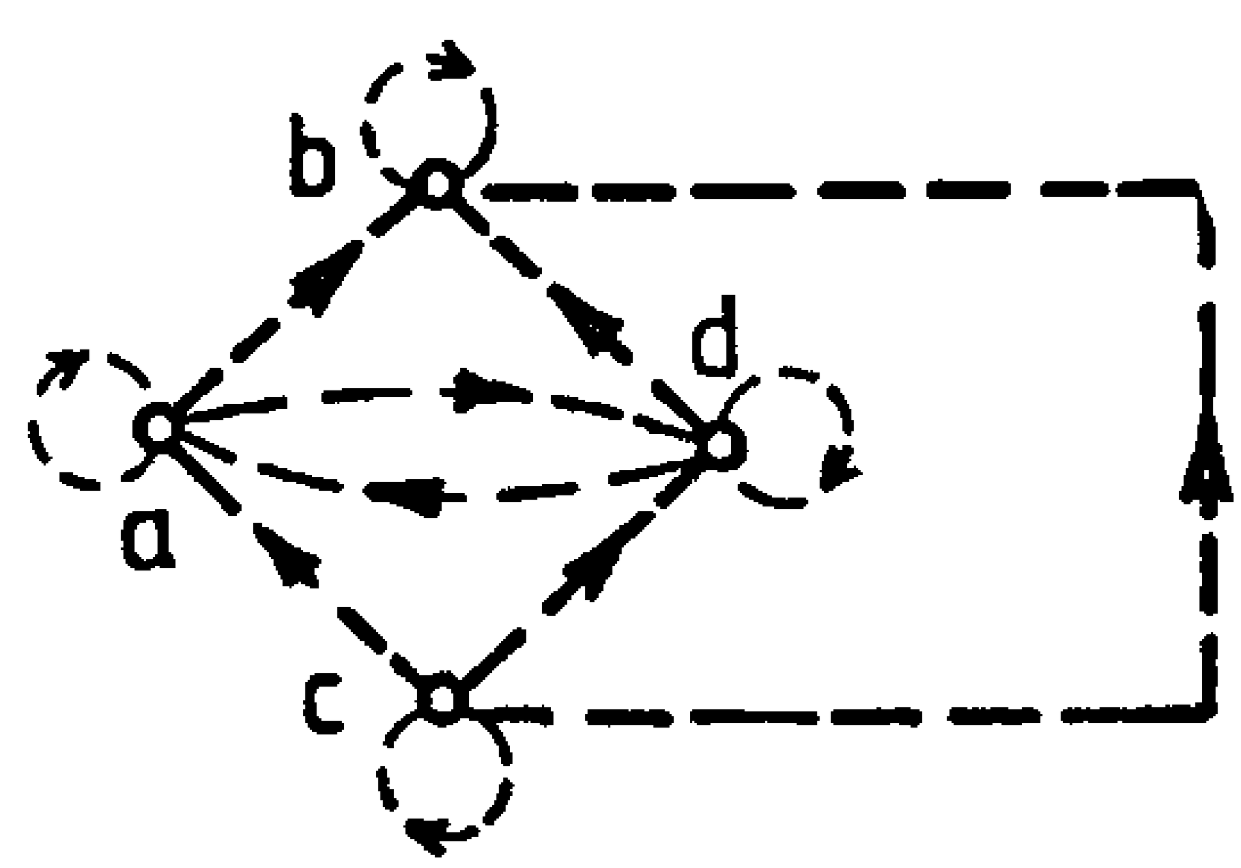


Figure 2. Complete order.

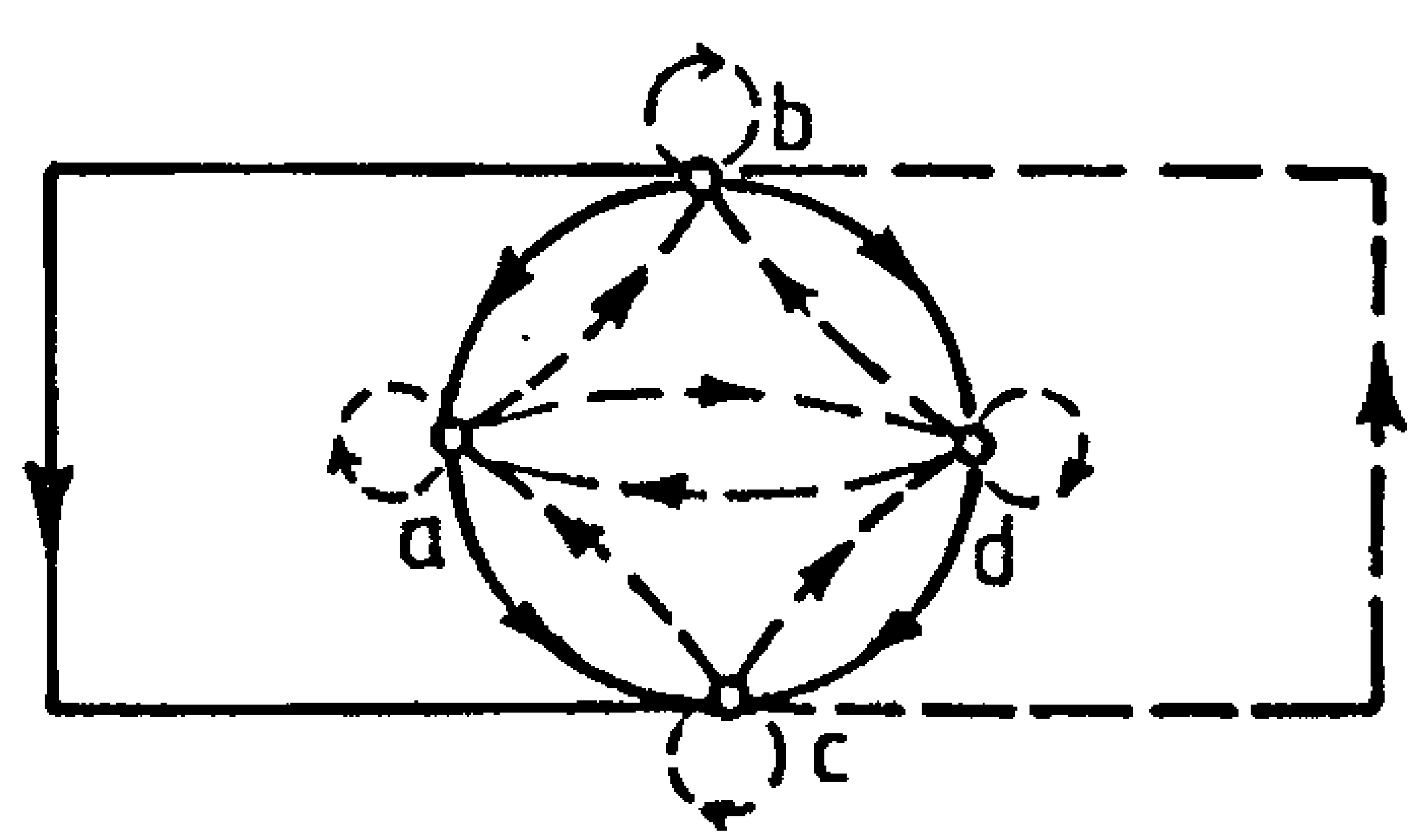


Figure 3. Chromatic graph.

(i) In a preference order the subgraph with continuous edges by itself satisfies transitivity, so does the subgraph with broken edges; (ii) Between any pair of distinct nodes there are exactly two edges and they are in the opposite directions, and (iii) Between any pair of nodes, if there is a continuous edge in one direction, the edge in the other direction is always broken i.e. we cannot have continuous arrows in both directions. We will use this fact later to obtain a corollary to Lemma 1.

Proof of Arrow's Dictator Theorem

Consider an election with three candidates and four voters. A voting system is a function $f(v_1, v_2, v_3, v_4)$ as shown in table 1.

Each variable v_i ranges over all the 13 different preference orders possible. Thus, the value of the function has to be specified for 13^4 different voting patterns to define the voting system (the value of the function also being one of the 13 preference orders). The general definition of a voting system should be clear from this example. A *voting* system is simply a function $f(v_1, v_2, \ldots, v_n)$, where all the v_i's are preference orders with the same number of nodes m (if there are m candidates). Obviously a given preference order of m nodes will have m^2 edges.

In our analysis we assume at least three candidates. An axiom assumed by Arrow in the statement of his theorem is the Axiom of Independence: The preference or nonpreference of the society for the ordered pair of candidates (x, y) depends only on the preference or nonpreference of the voters for that particular ordered pair.

Another axiom assumed by Arrow is the

Axiom of Unanimity: If all the voters vote in one way for an ordered pair of candidates, the voting system also votes the same way.

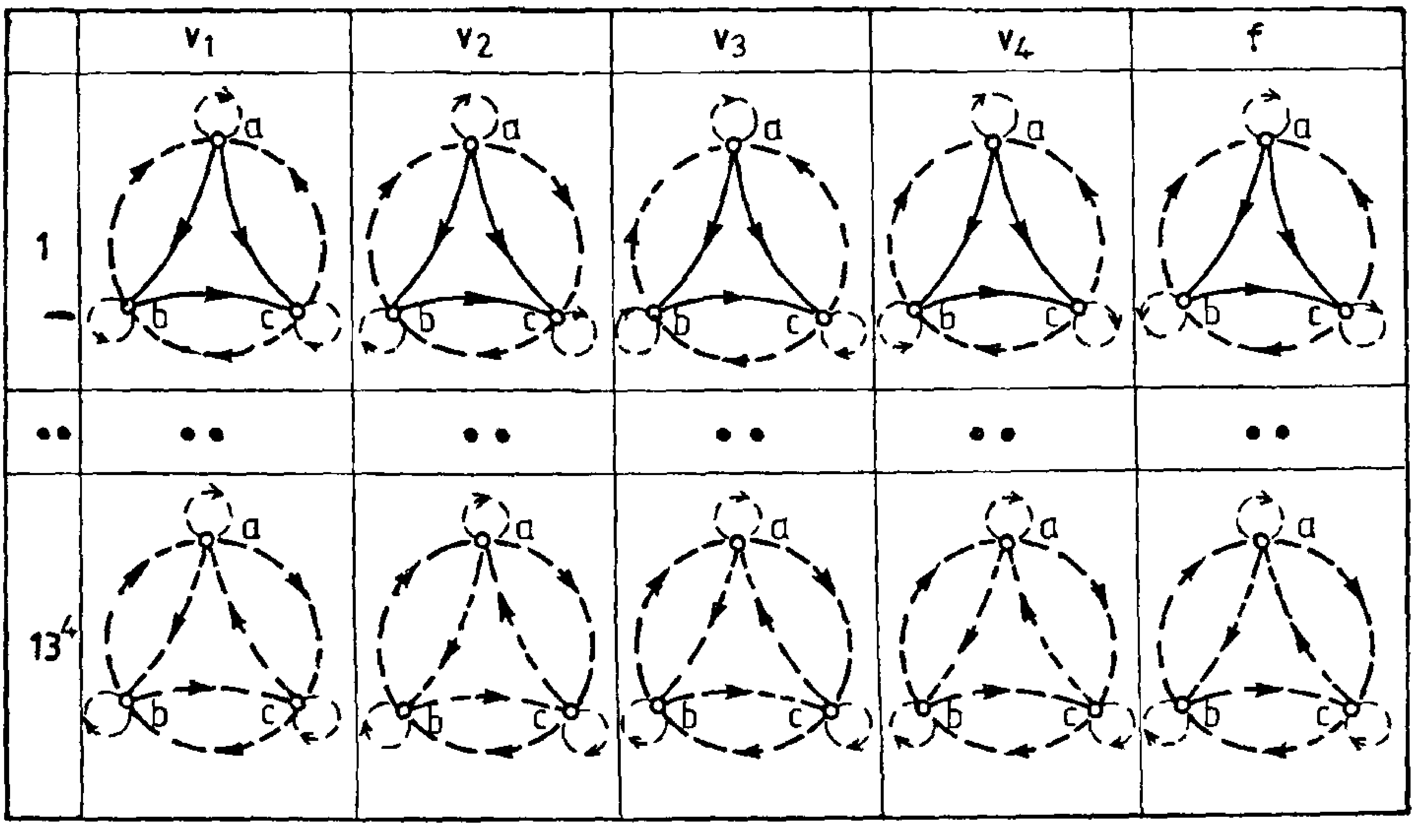
Arrow's Dictator Theorem: If a voting system satisfies the axioms of unanimity and independence then $f(v_1, v_2, \ldots v_n) = v_d$ where d is one of the integers in 1 to n.

We carry out the proof in two steps. We use the abbreviation xKy for the statement that the voter K has (strict) preference for the ordered pair (x, y) and the abbreviation xKy for nonpreference of K for the ordered pair (x, y). We use xK^*y for the statement that all voters other than K have voted nonpreference for the ordered pair (x, y). We use the symbol S for the voting system.

Lemma 1: There exist two specific candidates a and b and a particular voter D such that aDb and aD^*b implies aSb.

Proof: In the table for the voting system, we look at the continuous edges between any ordered pair of candidates in the graphs in the last column (corresponding to the choice of the voting system). For

Table 1



each individual continuous edge in each graph we look at the corresponding edges in the graphs pertaining to the voters on the left side. We count the total number of continuous edges in the corresponding positions paying attention to the direction of the edge also.

From the axiom of unanimity it should be clear that this number has to be between 1 and n inclusive. Of all these numbers we look for the minimum. We wish to show that the minimum is indeed one. We give a proof by contradiction. If the minimum is not one, let it be more than one and let the ordered node pair for which this occurs be (a, b). Classify these voters with continuous edges into two non-empty sets V_1 and V_2. Let the rest of the voters be called V_3. Let r be any candidate other than a and b. Consider a voting pattern of these three sets of voters as shown in figure 4.

Because of our assumption above, the graph of S will have to have a continuous edge from a to b, and broken edges from a to r and from r to b. But then from transitivity there has to be a broken edge from a to b. Thus the graph for S will have to be as shown in figure 4.

Notice that in the graph S there are two edges in the same direction between a and b and this is impossible in a preference order according to Fact (ii) mentioned earlier. Thus we conclude that the minimum could not have been more than one, and it has to be exactly equal to one. We will designate this unique voter by D. We have proved Lemma 1. A corollary immediately follows from Fact (iii) given earlier. Corollary: $b\bar{D}a$ implies bSa.

Lemma 2: For any pair of candidates x and y, and the voter D mentioned earlier, xDy implies xSy.

Proof: Let the set of voters other than D be designated by V and let x be any candidate other than a and b. Let the voting pattern of D and V be as shown in figure 5. Because of the axiom of unanimity the graph for S has to have a continuous edge from x to a and because of Lemma 1, a continuous edge from a to b. From transitivity it follows that the S graph must have an edge from x to b, as shown in figure 5.

Thus xDb implies xSb, where x is any candidate other than a and b. But then, it is a simple matter to see that xDb implies xSb, even if x happens to be a.

Consider the voting pattern as shown in figure 6

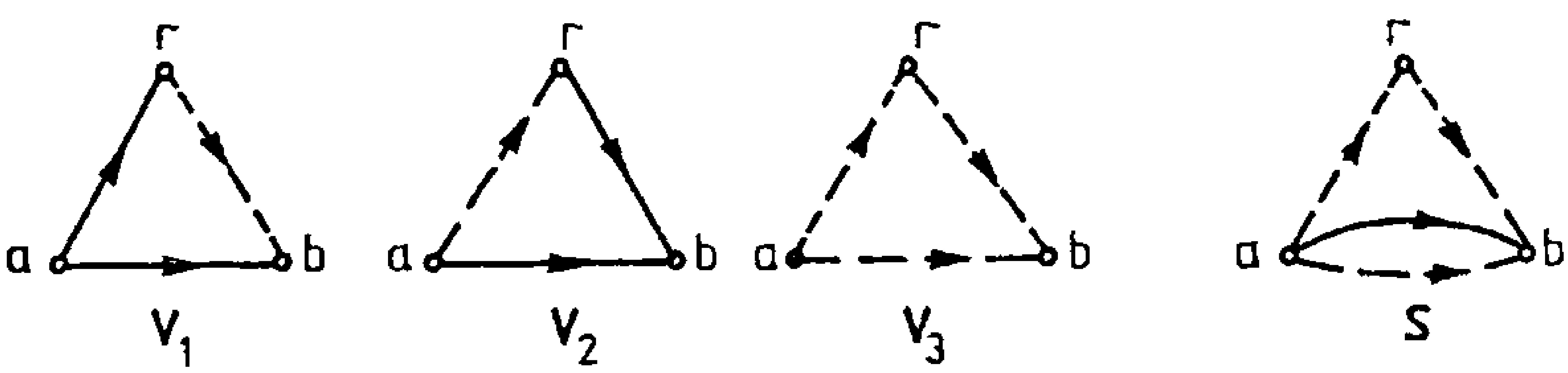


Figure 4. Emergence of the dictator.

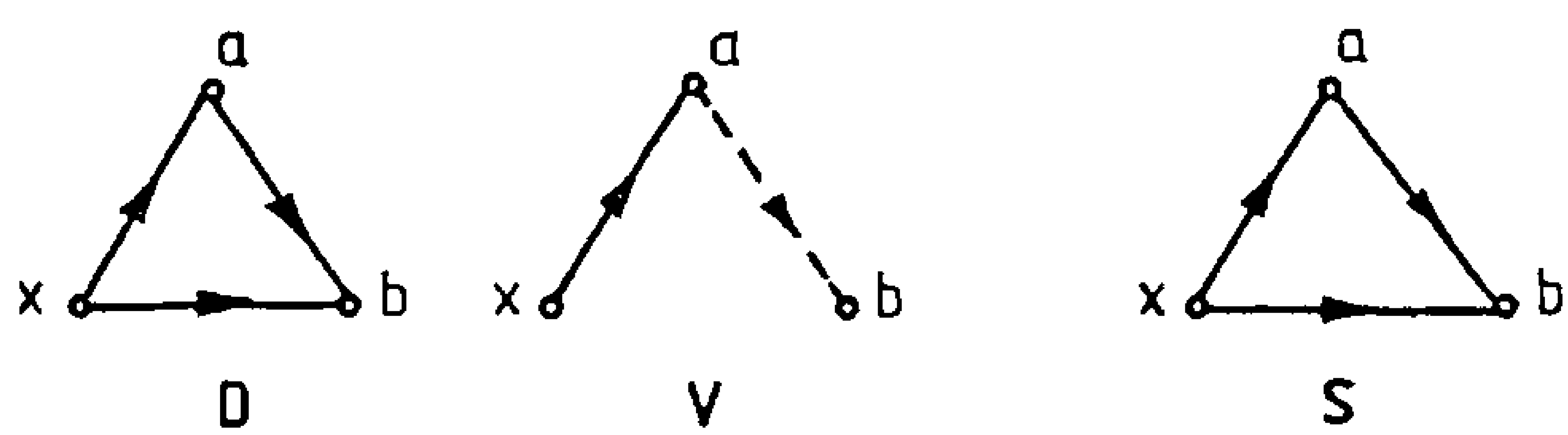


Figure 5. Consolidation by the dictator.

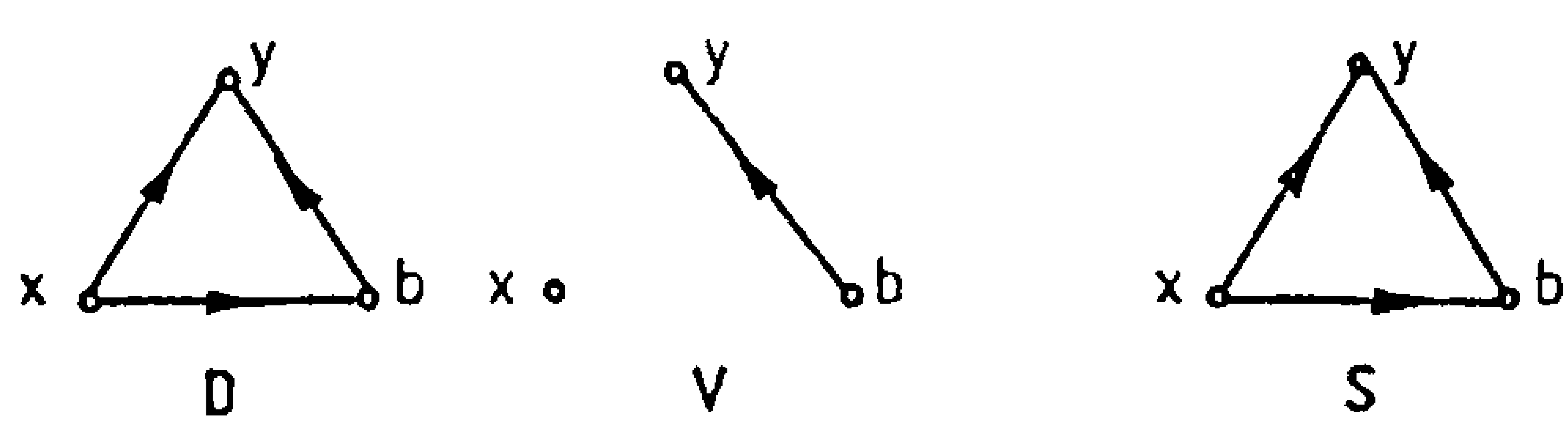


Figure 6. Supremacy of the dictator.

where x and y are candidates other than b.

As a consequence of the axiom of unanimity S graph must have a continuous edge from b to y and there must be a continuous edge from x to b because of our immediately preceding result. From transitivity, the continuous edge from x to y has to be present in the S graph. Thus xDy implies xSy, where x and y are two distinct candidates. We have proved Lemma 2.

We have shown that the continuous part of D graph is contained in the continuous part of S graph. Similarly it can be shown that the broken part of D graph is contained in the broken part of S graph. To carry out the proof it is only necessary to note the corollary to Lemma 1 and to switch continuous and broken edges in figures 5 and 6. Arrow's dictator theorem immediately follows.

Percentage Voting System

As a consequence of Arrow's theorem Paul Samuelson has stated the following (The New York Times, October 26, 1972): "Men have always sought ideal democracy–the perfect voting system.... But perfound minds have known for centuries the paradox of voting.... What is needed, you will think, is a new genius to invent a voting system.... What Kenneth Arrow proved once and for all is that there cannot possibly be found such an ideal voting scheme: The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for achimera, for a logical self-contradiction.... But it is no exaggeration to say that only a score of scholars were able to follow Arrow's early researches in these esoteric fields. Now scholars all over the world are engaged in trying to salvage what can be salvaged from Arrow's devastating discovery which is to mathematical politics what Godel's 1931 (theorem is to mathematical logic).

The simple proof of Arrow's theorem given above has been prompted by these remarks of Samuelson. Also, the question arises whether we should take Samuelson's statements literally and resign to the fact that democracies are not possible. Our intention is to show that the percentage voting system (PVS), being suggested here, clearly gives a way out of the dilemma. PVS can be simply stated as follows. Every voter is given 100 cent-votes instead of a single vote and it is up to the individual voter to distribute these 100 cent-votes to the candidates as he pleases. The candidate who gets the largest number of total cent-votes is declared the winner as usual.

To explain the motivation for the PVS, we will assume that one unit of *goodwill* is available to the society and the society would like to distribute it to the candidates in the most rational manner possible. To achieve this, the society first distributes the one unit of goodwill equally amongst all voters, consistent with the fact that most constitutions in the countries of the world insist on this. Each individual voter is presumed to distribute the goodwill available to him, amongst the candidates as he pleases, consistent with the fact that the freedom of expression is guaranteed by most constitutions of the world. The candidate who has collected the maximum goodwill is chosen as the winner. The PVS that we have suggested is obviously a method of ascertaining the goodwill collected by the candidates.

It is interesting to note that the goodwill collected by the candidates can be considered as a probability distribution and some statements about goodwill theory can be made as follows. Let the number of candidates be m and the number of voters be n, and let p_{ij} be the goodwill obtained by the ith candidate from the jth voter. Let p_i be the total goodwill collected by the ith candidate. Then the following definitions seem to be meaningful.

The *popularity* of the ith candidate

$$P_i = \log(p_i/q), \text{ where } q = 1/m.$$

The *polarization* P, of the constituency i.e. the tendency of the voters to get attached to individual candidates, can be defined as

$$P = \sum_{i=1}^{m} p_i \log\left(\frac{p_i}{q}\right)$$

Further we can define a *leader* of the society as one whose popularity is at least equal to the polarization. These concepts are obviously borrowed from Shannon's definition of entropy and hence we shall not elaborate on it except to state that polarization will always be non-negative.

A widely discussed issue about voting systems is the case when some candidates get disqualified after the voting. In the percentage voting system all that we have to do in such an event is to redistribute the goodwill from a voter to the surviving candidates in the proportion in which they got the goodwill in the original voting. The total goodwill for each candidate is added as before and winner declared. In the event every one of the surviving candidates has zero goodwill from a particular voter, then such a voter's goodwill is totally ignored.

To circumvent Arrow's paradox with respect to our Presidential election in India (see Open Page, THE HINDU, 18 September 1984), we can adopt the following procedure. Instead of asking the Members of Parliament to give the entire value of their vote in favour of a preference order, they should be allowed to split the value of their votes and distribute them to the different candidates according to each individual voter's wishes. As usual the candidate who gets the highest number of votes wins.

CONCLUSION

From what we have said above, it should be clear that the PVS satisfies the two most crucial axioms of modern constitutions, viz., (i) equality of individuals, (ii) freedom of expression for individuals. The most significant factor we have to recognize here is that, the preference distribution used in PVS gives more freedom to the voter than the preference order that is currently being used around the world.

22 April 1987

1. Arrow, K. *Social choice and individual values*, Wiley, New York, 1963.
2. Sen, A. K., *Collective choice and social welfare*, Holden-Day, San Francisco, 1971.

NEWS

DST WORKSHOP ON 'BIOSYSTEMATICS OF INSECTS OF IMPORTANCE IN AGRICULTURE, MEDICINE AND FORESTRY'

A DST-sponsored Workshop on the above theme was conducted from 27-30th April with senior Entomoligists from nearly 20 universities and an equal number of young scientists. Inaugurating the workshop, Prof. S. Krishnaswamy, Vice-Chancellor, Madurai Kamaraj University, exhorted the participants to profitably use the emerging techniques in biosystematic studies so as to have a better and proper understanding of species. The twenty five papers presented, related to the role of ultrastruture, karyology, bio-chemical parameters, ecobehaviour and biogeography, sufficiently emphasised the need for such an integrated approach in order to be able to meaningfully assess the increasing variations in the natural population of insects of Agricultural, Medical and Forestry importance, more noticeably in such pests species or vector species tending to exhibit what has come to be known as 'Biotypes' 'siblings' etc. Of particular interest were the special lectures on 'Molecular Biology and Biosystematics of Insects' by Prof. Kunthala Jayaraman of the Anna University; 'LDH system as a tool in Biochemical Systematics' by Prof. Kamalakar Rao of the Pachaiyappa's College, Madras and 'Raciation in *Drosophila* as demonstrated by laboratory experiments' by Dr Ranganath of the Mysore University, which discussed the emerging trends in the field of Biosystematics. The plenary lecture by Prof. T. N. Ananthakrishnan of the Entomology Research Institute on 'The Dimensions of Species' highlighted the need for indepth investigations on various aspects involving diverse methodologies, to have a meaningful understanding of the concept of speciation, more particularly in view of the dynamics of the species.

Demonstration sessions on methodologies involving ultrastructure study. Electrophoretic studies for LDH and proteins, Karyology etc were also included.