BOUNDS ON THE COSMOLOGICAL CONSTANT

P. S. JOSHI and S. M. CHITRE
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.

ABSTRACT

Without appealing to the exact homogeneity and isotropy of the universe, bounds on the cosmological constant are derived in the framework of a general spacetime. The special case of Friedmann cosmologies is also discussed.

INTRODUCTION

THE consequences of a non-zero cosmological constant in the context of the dynamics of the universe have been extensively discussed[1,2]. In fact, a universal constant Λ can be inserted in Einstein's equations without destroying the general covariance of the theory[3] to write,

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi G T_{ij} \tag{1}$$

In recent years, there has been a tendency[2] to regard the cosmological constant as one of the most fundamental physical entities. It is generally accepted that the cosmological constant should be considered as part of the stress-energy tensor representing the non-zero vacuum expectation value of T_{ij} generated by quantum fluctuations. Thus, Λ is a number (with units cm^{-2}) which can, in principle, be calculated from local quantized fields[4,5].

On the other hand, the observational methods to determine Λ invoke the departures from the exact Hubble flow for the distant galaxies. The bounds thereby determined for Λ are very tight ($|\Lambda| < 10^{-55}\,cm^{-2}$)[2]. It should be noted, however, that the above method is beset with several stringent assumptions regarding the universe *as a whole* (whereas we can only observe part of it), namely, its *exact homogeneity and isotropy* and *an exact measurement of the Hubble constant* which is very uncertain at the moment. It would, therefore, be highly desirable to have an alternative approach to this problem where the above assumptions can be relaxed. It is the purpose of this paper to present such an alternative method to place limits on Λ where we neither assume the global homogeneity and isotropy of the universe, nor the values of the Hubble constant H_o.

THE GENERAL FORMALISM

In our cosmological consideration, we merely assume for the global geometry that the spacetime admits a foliation[6] by spacelike hypersurfaces. This is a property shared by all the physically reasonable cosmological models such as Friedmann or steady-state. Two further conditions that we impose are that Einstein's equations (1) with a cosmological constant hold, and that there are no modes carrying negative energy[7].

The world-lines of galaxies or other material particles are represented by the non-spacelike trajectories in our model. We examine[8] the evolution of these nonspacelike paths from the present epoch S_o into the past. The gravitational focussing effects[7] generated by the stress-energy density are characterized by the expansion parameter θ of the congruence of timelike geodesics which are orthogonal to S_o and which obey the Raychaudhuri equation,

$$d^2x/dt^2 + F(t)x = 0, \tag{2}$$

where x is defined by the equation $\theta = 1/x\,(dx/dt)$ and

$$F(t) = \tfrac{1}{3}(R_{ij}V^iV^j + 2\sigma^2). \tag{3}$$

The quantity σ represents the shear of the congruence.

There is, however, a constraint on the occurrence of the focal or conjugate points which is, given any two causally connected events, there exists a maximal geodesic joining the two which can contain no conjugate points. The zeros of (2) represent the conjugate points which result from the gravitational focussing of non-spacelike trajectories. When $F(t) > \lambda^2 > 0$, the maximal possible extension into the past of any non-spacelike curve from S_o is given by $\frac{\pi}{2}(3/\lambda)^{\frac{1}{2}}$. We compute $F(t)$ using (1) to obtain,

$$R_{ij}V^iV^j > 4\pi G\rho - \Lambda c^2, \tag{4}$$

This gives for the maximum possible age t_{max} in the past:

$$t_{max} = \frac{\pi}{2}\left(\frac{3}{4\pi G\rho - c^2\Lambda}\right)^{1/2}. \tag{5}$$

It should be noted that departures from spherical symmetry or perturbations from exact homogeneity and isotropy are permitted in the analysis.

BOUNDS ON THE COSMOLOGICAL CONSTANT

We note that observations on the departure from the Hubble law for distant galaxies mentioned earlier only imply that $|\Lambda|$ is very small, and that, Λ may have a positive or negative sign. We consider here each case separately.

For obtaining limits when Λ is negative, we can ignore the contribution from ρ-term in (5) (including the same would, in any case, tighten the bounds given by us). Then (5) can be written as

$$t_{max} < \frac{\pi}{2}\left(\frac{3}{|\Lambda| c^2}\right)^{1/2}. \quad (6)$$

Now, a variety of considerations[9–12] independent of cosmological models, such as the ages of stars and globular cluster ages place the lower limit to the age of universe in the range $(8\text{–}24)\times 10^9$ years. Clearly, we must have $t_{ob} < t_{max}$, and (6) then gives,

$$|\Lambda| = (3\pi^2/4t^2_{max}c^2) < (3\pi^2/4t^2_{ob}). \quad (7)$$

Thus, for example, with $t_{ob} = 20\times 10^9$ years, we have,

$$|\Lambda|_{max} = 2.1\times 10^{-56}\ \text{cm}^{-2}. \quad (8)$$

We note that for the entire range of ages $(10\text{–}24\times 10^9$ years), $|\Lambda|_{max}$ turns out to be of the order of $10^{-56}\ \text{cm}^{-2}$. Hence, the limits computed here within a general framework turn out to be actually better than those ($10^{-55}\ \text{cm}^{-2}$) obtained using the observational method. In fact, this is not unexpected in view of the large uncertainties prevailing today in the measurements of the Hubble constant H_o. When Λ is positive, (5) implies that

$$\Lambda < (4\pi G\rho/c^2) \quad (9)$$

Also, we have,

$$4\pi G\rho - \Lambda c^2 = (3\pi^2/4t^2_{max}) \leqslant (3\pi^2/4t^2_{ob}). \quad (10)$$

A combination of (9) and (10) limits the range of Λ to

$$4\pi G\rho - (3\pi^2/4t^2_{ob}) \leqslant \Lambda c^2 < 4\pi G\rho. \quad (11)$$

With the input of data on the presently measured energy densities, the RHS of (11) provides tight upper limits on Λ. For example, for $\rho = 5\times 10^{-29}\ \text{g cm}^{-3}$, we get $\Lambda < 4.7\times 10^{-56}\ \text{cm}^{-2}$. For lower limits the situation here is not equally clear as one requires the data on observed ages as well as that on densities. Thus, for example, for $\rho = 10^{-30}\ \text{g cm}^{-3}$ and $t_{ob} = 20\times 10^9$ years, the lower limits are negative; which is not very useful.

Finally, we examine the bounds on the cosmological constant in the framework of the Friedmann model. Since this is a special case of the general situation discussed above a substitution of the energy density $\rho_o = 3q_oH_o^2/4\pi G$ in (5) gives

$$t_o \leqslant t_{max} = \frac{\pi}{2}\left(\frac{3}{3q_oH_o^2 - \Lambda c^2}\right)^{1/2}. \quad (12)$$

Here q_o is the usually defined present value of the deceleration parameter. Again, using the Friedmann relation $t_o = H_o^{-1} f(q_o)$ in (12), we obtain,

$$\Lambda c^2 \geqslant [3q_o[f(q_o)]^2/t_o^2] - (3\pi^2/4t_o^2). \quad (13)$$

The function $q_o[f(q_o)]^2$ has an interesting property: As q_o tends to infinity, $q_o^{1/2} f(q_o)$ has a maximum value $\pi/2\sqrt{2}$. Using this we get,

$$\Lambda c^2 \geqslant -3\pi^2/8t_o^2. \quad (14)$$

Another constraint that follows from (12) is $\Lambda c^2 < 3q_oH_o^2$. The above limiting procedure then again gives,

$$\Lambda < 3\pi^2/(8c^2t_o^2). \quad (15)$$

Thus, independently of the sign of the cosmological constant, we have,

$$|\Lambda|_{max} = 3\pi^2/(8c^2t_o^2). \quad (16)$$

A typical value for the age of the universe, $t_o = 2\times 10^{10}$ years then gives $|\Lambda|_{max} = 1.04\times 10^{-56}\text{cm}^{-2}$. The observations on ages thus again provide relativity stringent limits on the value of the cosmological constant within the Friedmann cosmologies.

PARTICLE MASS UPPERLIMITS WHEN $\Lambda \neq 0$

The cosmological considerations have been very successful in placing stringent limits on the masses of elementary particles, notably those of neutrinos, leptons etc[13]. It was, however, pointed out by Barrow[1] that most of these results rest on the unverified cosmological assumption of $\Lambda=0$. It is extremely likely that a non-zero value of Λ will emerge as a result of phase transitions in the early universe. Barrow has stressed that in such a situation, the particle mass upperlimits will stand revised considerably as argued below. The assumptions of complete homogeneity and isotropy give the Friedmann equation,

$$(\dot{R}^2/R^2)=(8\pi G\rho/3)-(k/R^2)+\Lambda. \quad (17)$$

With the usual definitions of the Hubble constant H_o, deceleration parameter q_o and dimensionless density parameter $\Omega_o=8\pi G\rho_o/H_o^2$, we can write,

$$\Omega_o=(2\Lambda/3H_o^2)+2q_o. \quad (18)$$

Hence, if $\Lambda > 0$, the matter density can be considerably higher without adversely affecting the value of q_o and particle masses can be correspondingly larger.

Now, the observational data on the lower limits for ages of stars, globular clusters, the red shift magnitude diagram and the quasar data imply that when $\Lambda\neq 0$, the parameters q_o and Ω_o obey the following limits:

$$-4.4<q_o<5.6,\ 0.05<\Omega_o<9.4. \quad (19)$$

In such a situation, we have an upperlimit on ρ_o:

$$\rho_o<1{\cdot}7\times 10^{-28}\,h_o^2\,\mathrm{g\,cm^{-3}}. \quad (20)$$

This weakens the particle mass limits considerably. Taking the case of light neutrinos, we have

$$m_\nu \lesssim 280\,h_o^2\,\mathrm{eV}, \quad (21)$$

instead of

$$m_\nu \lesssim 60\,h_o^2\,\mathrm{eV},\ \text{for the } \Lambda = 0 \text{ case}. \quad (22)$$

The bounds for heavy neutrinos, heavy leptons etc will be similarly revised.

We can now use the global consideration given above to further remove the unverified cosmological idealization of exact homogeneity and isotropy towards obtaining the revised particle mass limits when $\Lambda\neq 0$. Again using (5) and $t_{ob}<t_{max}$, we can write:

$$\rho \leqslant \left\{ \frac{3\pi^2}{4t_{ob}^2} + \Lambda c^2 \right\} \frac{1}{4\pi G}. \quad (23)$$

The input of observed ages would again provide upper limits on the total energy density for a cosmological particle continuum. For example, taking $t_{ob}=2\times 10^{10}$ years, we get

$$m_\nu \lesssim 73\,\mathrm{eV}, \quad (24)$$

for $\Lambda=2\times 10^{-56}\,\mathrm{cm^{-2}}$. Thus, it turns out that a non-zero cosmological constant significantly changes the elementary particle mass constraints in cosmology. Clearly, the case for a non-zero value of the cosmological constant Λ is still open. We have attempted to derive bounds on the value of Λ (both positive and negative) using the data available on the lower limits of ages of oldest objects in the universe. We have also examined the effect of non-zero Λ on the upper limits of particle masses.

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