INTEGER SOLUTIONS OF SOME EQUATIONS

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ABSTRACT

An account of author's recent work on perfect powers in values of certain polynomials at integer points is given.

INTRODUCTION

We shall consider four equations. Though they look different, there is much in common the way in which the basic results are applied for investigating them. Among the basic results, there is a theorem of Baker[1] on the approximations of certain algebraic numbers by rationals and an estimate of Baker[2] on linear forms in logarithms. This is, mainly, an account of author's recent work and no attempt has been made to give a systematic survey of the related results considered in this article. For this, we refer to chapters 10 and 12 of Shorey and Tijdeman[3].

Let us write

$$1+3+3^2+3^3+3^4 = 121 = 11^2,$$
$$1+7+7^2+7^3 = 400 = 20^2,$$
$$1+18+18^2 = 343 = 7^3.$$

These numbers satisfy the following two properties.

(a) The number 121 has all the digits equal to one with respect to the base 3, the number 400 has all the digits equal to one with respect to the base 7 and the number 343 has all the digits equal to one with respect to the base 18.

(b) The numbers 121, 400 and 343 are all perfect powers (= 4, 8, 9, 16, . . .).

This leads us to consider the question of determining perfect powers in the set of integers with all the digits equal to one with respect to a given base. For an integer a with $1 < a < 10$, Obläth[4] completely determined the set of perfect powers with all the digits equal to a in their decimal expansions. He showed that 4, 8 and 9 are the only integers with this property. The case $a = 1$ remains uncovered in Obläth's result. In this direction, Shorey and Tijdeman[5] proved that for a fixed integer $x > 1$, say $x = 10$, there are only finitely many perfect powers with all the digits equal to one in their expansions with respect to the base x. For a sufficiently large integer x, the author[6] showed that the number of qth perfect powers $(= 2^q, 3^q, \ldots$ and $q > 1)$ whose digits satisfy the above property is less than q.

An integer with all the digits equal to one with respect to the base x is of the form

$$1+x+\ldots+x^{n-1} = (x^n-1)/(x-1).$$

Thus the above results can be interpreted in terms of the integer solutions of the following equation

$$y^q = \frac{x^n-1}{x-1} \text{ in integers } x > 1, y > 1, q > 1, n > 2. \tag{1}$$

Ljunggren[7] proved that equation (1) with $q = 2$ has no solution other than the ones mentioned in the beginning of this section. One would like to show that (1) has only finitely many solutions in all the four variables x, y, q and n. For proving this conjecture, it suffices to show that (1) has only finitely many solutions if n is restricted to prime powers. In other words, it suffices to show that (1) has only finitely many solutions if n is restricted to $\omega(n) = 1$ where $\omega(n)$ denotes the number of distinct prime factors of n. See Shorey[6,8] where it was proved that (1) has only finitely many solutions in all the four variables if n is restricted to $\omega(n) > q-2$. A positive integer which has no divisor other than one and itself is called a prime number.

Goormaghtigh observed that

$$31 = \frac{2^5 - 1}{2 - 1} = \frac{5^3 - 1}{5 - 1},$$

and

$$8191 = \frac{2^{13} - 1}{2 - 1} = \frac{90^3 - 1}{90 - 1}.$$

Thus 31 has all the digits equal to one with respect to the base 2 as well as the base 5. Similarly 8191 has all the digits equal to one with respect to the base 2 as well as the base 90. He conjectured that these are the only ones that have all the digits equal to one with respect to two distinct bases. Shorey[9] showed that there are at most 17 integers which have all the digits equal to one with respect to two given distinct bases, say x and y. In other words, there are at most 17 pairs (m, n) of positive integers satisfying

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}. \tag{2}$$

Again one would like to show that this equation has only finitely many solutions in integers $x > 1, y > 1, m > 2$ and $n > 2$. It follows from the theorems of Thue[10], Siegel[11] and Davenport, *et al*[12] that (2) has only finitely many solutions if any two of the four variables x, y, m and n are fixed. This is also the case if xy is composed of primes from a given set or $|x - y|$ is bounded. These assertions are due to Balasubramanian and Shorey[13] and Shorey[14].

Let $A > 0$, $B > 0$ and $k \neq 0$ be integers. We consider Pillai's equation

$$Ax^m - By^n = k. \tag{3}$$

Pillai[15] conjectured that (3) has only finitely many solutions in integers x, y, m and n (all > 1) with $mn \geqslant 6$. In view of the existence of infinitely many integer solutions of Pell's equations, the restriction $mn \geqslant 6$ is necessary in the above conjecture. By the results of Baker[16] and Schinzel and Tijdeman[17], Pillai's conjecture is confirmed if any one of the four variables x, y, m and n is fixed. For given integers $A > 0$, $B > 0$, $x \geqslant 4$ and $y \geqslant 4$, Shorey[9] showed that there are at most 9 pairs (m, n) of positive integers satisfying (3) with $k = 1$. If $A = B = k = 1$, LeVeque[18] proved the above result of the author with 9 replaced by 1.

Equation (3) with $A = B = k = 1$

$$x^m - y^n = 1, \tag{4}$$

is the equation of Catalan. In fact Catalan[19] conjectured that (4) has only one solution given by

$$x = 3,\ m = 2,\ y = 2,\ n = 3.$$

In other words, Catalan conjectured that 9 and 8 are the only perfect powers that differ by one. Tijdeman[20] proved, in principle, the conjecture of Catalan. This means that if x, y, m and n (all > 1) are integers satisfying (4), then the maximum of x, y, m and n is bounded by a computable absolute constant. However this constant turns out to be so large that finitely many remaining cases can not be checked on a computer to confirm the conjecture of Catalan.

By consecutive integers, we shall always mean consecutive positive integers. It is easy to see that the product of two consecutive integers is never a perfect power. Erdös[21] and Rigge[22] independently proved that the product of two or more consecutive integers is never a square. Further Erdös and Selfridge[23], developing on the method of Erdös[24], confirmed a very old conjecture by proving that the product of two or more consecutive integers is never a perfect power. We consider a more general question, namely perfect powers in products of integers from a block of consecutive integers.

For an integer $v > 1$, we denote by $P(v)$ the greatest prime factor of v and we write $P(1) = 1$. Let $m \geqslant 0$ and $k \geqslant 2$ be integers. Let $d_1, \ldots, d_t$ with $t \geqslant 2$ be distinct integers lying between 1 and k. For positive integers b, l and y with $l \geqslant 2$ and $P(b) \leqslant k$, we consider the equation

$$(m + d_1) \ldots (m + d_t) = by^l. \tag{5}$$

If the greatest prime factor of the left hand side of (5) is at most k, then (5) is satisfied with $y = 1$ and $b = (m + d_1) \ldots (m + d_t)$. Let p be a prime greater than k dividing $m + d_i$ for some i between 1 and t. Since a prime greater than k can divide at most one integer among k consecutive integers, we see from (5) that $m + d_i$ is, in fact, divisible by p^l.

Therefore

$$(k+1)^l \leqslant p^l \leqslant m+d_i \leqslant m+k,$$

which implies that

$$k^l < m. \tag{6}$$

From now onward, we assume that (6) is satisfied.

For $l \geqslant 3$, we put

$$v_l = \frac{1}{2}\left(1 + \frac{4l^2 - 8l + 7}{2(l-1)(2l^2 - 5l + 4)}\right).$$

We observe that

$$v_l \leqslant 7/8, \quad l = 3, 4, \ldots.$$

The present author[8, 25] proved that equation (5) with

$$l \geqslant 3, t \geqslant v_l k, \tag{7}$$

implies that k is bounded by an absolute constant. Thus, if k exceeds a sufficiently large absolute constant, any product of at least $(7/8)k$ distinct integers among k consecutive integers $m+1, \ldots, m+k$ is never a cube or a higher power. For large values of l, the restriction (7) can be relaxed considerably[8]. For example, (5) implies that $t/k \to 0$ whenever both k and l tend to infinity. According to a conjecture of Erdös and Turk[26], this assertion cannot be strengthened to

$$t/k \to 0 \quad \text{whenever} \quad k \to \infty.$$

Apart from the results of Baker mentioned in the beginning of this article, the best possible estimates for linear forms in logarithms (with α_i's close to one) are crucial in the proofs of these results. The proofs also depend on the method of Roth[27] as elaborated in Halberstam and Roth[28] on difference between consecutive v-free integers. For squares, much weaker results are available. For $\varepsilon > 0$ and k exceeding sufficiently large number depending only on ε, the present author[25] proved that any product of

$$k - (1-\varepsilon)k\,\frac{\log\log k}{\log k}$$

distinct integers among k consecutive integers $m+1, \ldots, m+k$ is never a square. The proof depends on a theorem of Baker[16] on integer solutions of a hyperelliptic equation and lemma 4 of Erdös[24].

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