CLASSICAL LIMIT OF THE TRAJECTORIES OF SPIN-$\frac{1}{2}$ PARTICLES IN GRAVITATIONAL FIELD

RAMESH CHANDRA

Department of Mathematics, St. Andrew's College, Gorakhpur 273001, India.

ABSTRACT

The path of a spin-$\frac{1}{2}$ particles in classical limit has been obtained via the WKB approximation of the energy-momentum tensor of the Dirac field.

INTRODUCTION

In general theory of relativity there are three usual methods for the determination of the equation of motion of a free particle (classical): (i) the geodesic of the space-time geometry, (ii) the conservation law and (iii) the variational principle. The method (i) is based on infinitesimal version of the law of inertia which is valid in a local Minkowskian space-time and the equality of the inertial and gravitational masses of a particle[1]. The method (ii) is based on the conservation law for dust fluid[2] and the method (iii) is based on the variation of the action integral

$$-\int m ds, \tag{1}$$

where m is the gravitational mass of the particle and ds is the metric of the space-time[3]. The action function $m ds$ is derived from the energy of the particle.

The significance of the motion of an elementary particle has been realized recently in neutron interferometer experiment[4]. Because of the fundamental difference between elementary particles and classical particles, the motion of an elementary particle cannot be derived either of the methods (i), (ii) and (iii). It is still a problem how to derive the motion of an elementary particle.

Recently, Audretsch derived the path of a spin-$\frac{1}{2}$ particle in classical limit via the WKB approximation of the Dirac field equation[5]. According to this procedure it has been shown that in the classical limit the path of a spin-$\frac{1}{2}$ particle follows the geodesic of the space-time geometry.

In this note we propose another method for the determination of the path of a spin-$\frac{1}{2}$ particle in classical limit via the WKB approximation of the energy-momentum tensor of the Dirac field.

From the conservation law of the energy-momentum tensor for a dust fluid it has been shown that the path of a dust particle follows the geodesic of the space-time geometry[2]. Also, in this procedure we are able to determine the conservation of the momentum density of the dust fluid. Therefore, from the classical limit of the conservation law for the energy-momentum tensor of the Dirac field not only the determination of the path of spin-$\frac{1}{2}$ particles in the classical limit but also, the conservation of the momentum density of the Dirac field in the classical limit is possible to determine.

The Lagrangian density for the Dirac field is given by

$$L = \sqrt{-g}\,\frac{\hbar}{2}\left[-i\bar{\psi}\gamma^{\mu}\psi_{;\mu} + i\bar{\psi}_{;\mu}\gamma^{\mu}\psi + \frac{2m}{\hbar}\bar{\psi}\psi\right], \tag{2}$$

where $-g$ is the determinant of the metric tensor $g_{\mu\nu}$, $\hbar$ the Plank constant, ψ and $\bar{\psi}$ the four spinor and its conjugate, m the mass of the spin-$\frac{1}{2}$ particle. γ^{μ} is defined as

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}. \tag{3}$$

The covariant derivative of the spinor is defined as

$$\psi_{;\mu} = \psi_{j\mu} + \Gamma_{\mu}\psi, \bar{\psi}_{;\mu} = \bar{\psi}_{j\mu} - \bar{\psi}\Gamma_{\mu}, \tag{4}$$

where the symbol 'j' denotes the partial derivative. Γ_{μ} are the Fock-Ivanenko coefficients. These are uniquely determined up to an additive multiple of unit matrix from the relation[6]

$$\gamma_{\mu;\nu} = \gamma_{\mu,\nu} - \gamma_{\varepsilon}\left\{{\varepsilon \atop \mu\nu}\right\} - \Gamma_{\nu}\gamma_{\mu} + \gamma_{\mu}\Gamma_{\nu} = 0 \tag{5}$$

$\left\{{\varepsilon \atop \mu\nu}\right\}$ are Christoffel symbols formed from the metric tensor $g_{\mu\nu}$. By varying the Lagrangian density (2) with respect to $\bar{\psi}$ and ψ separately, we get

$$i\hbar\gamma^{\mu}\psi_{;\mu} - m\psi = 0, \tag{6}$$

and

$$i\hbar\bar{\psi}_{;\mu}\gamma^{\mu} + m\bar{\psi} = 0. \tag{7}$$

The probability current J^{μ} is defined as

$$J^{\mu} = \bar{\psi}\gamma^{\mu}\psi \tag{8}$$

which is divergence free

$$J^{\mu}_{;\mu} = 0 \tag{9}$$

by virtue of (6) and (7). According to the Gordon decomposition of the probability current[7], we get

$$J^{\mu} = J^{\mu}_{c} + J^{\mu}_{M}, \qquad (10)$$

where

$$J^{\mu}_{c} = \frac{\hbar}{2m}(\bar{\psi}^{j\mu}\psi - \bar{\psi}\psi^{j\mu}), \qquad (11)$$

and

$$J^{\mu}_{M} = \frac{\hbar}{2m}(\bar{\psi}\sigma^{\mu\nu}\psi)_{j\nu}$$

$$= \frac{\hbar}{2m}(\bar{\psi}_{j\nu}\sigma^{\mu\nu}\psi + \bar{\psi}\sigma^{\mu\nu}\psi_{j\nu}), \qquad (12)$$

$$2\sigma^{\mu\nu} = i(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}). \qquad (13)$$

Both currents J^{μ}_{c} and J^{μ}_{M} are divergence free

$$J^{\mu}_{cj\mu} = 0, \qquad J^{\mu}_{Mj\mu} = 0. \qquad (14)$$

The energy-momentum tensor for the Dirac field is defined as[8]

$$\int T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\mathrm{d}\Omega = \delta\int L\,\mathrm{d}\Omega, \qquad (15)$$

where $\mathrm{d}\Omega$ is the infinitesimal four volume. Following Pauli[8], we have

$$\delta\gamma^{\mu} = \tfrac{1}{2}\gamma_{a}\delta g^{a\mu}, \qquad (16)$$

where

$$\gamma^{\mu} = e^{\mu}_{a}\gamma^{a}, e^{\mu}_{a}e^{\nu}_{b}g_{\mu\nu} = \eta_{ab}, \qquad (17)$$

e^{μ}_{a} are the orthotetrads, γ^{a} the standard Pauli matrices, and η_{ab} the Minkowskian metric tensor. The variation of γ^{μ} when combined with the variation of (15), we get

$$\delta\Gamma_{\mu} = \frac{i}{8}(g_{\nu\sigma}\delta\{^{\sigma}_{\mu\varepsilon}\} - g_{\varepsilon\sigma}\delta\{^{\sigma}_{\mu\nu}\})\sigma^{\nu\varepsilon}. \qquad (18)$$

The variations (16) and (18) are used to evaluate the variation of the Lagrangian density in (15) as follows

$$\int T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\mathrm{d}\Omega = \frac{\hbar}{2}\int\Big[-i\bar{\psi}\delta\gamma^{\varepsilon}\psi_{j\varepsilon} - i\bar{\psi}\gamma^{\varepsilon}\delta\Gamma_{\varepsilon}\psi + i\bar{\psi}_{j\varepsilon}\delta\gamma^{\varepsilon}\psi + i\delta\Gamma_{\varepsilon}\bar{\psi}\gamma^{\varepsilon}\psi\Big]\sqrt{-g}\,\mathrm{d}\Omega$$

$$= \frac{\hbar}{2}\int\Big[(-i\bar{\psi}\gamma_{(\nu}\psi_{j\mu)} + i\bar{\psi}_{j(\mu}\gamma_{\nu)}\psi)\delta g^{\mu\nu} + \tfrac{1}{2}(g^{\rho\nu}J^{\mu} - g^{\rho\mu}J^{\nu})(g_{\mu\sigma}\delta\{^{\sigma}_{\nu\rho}\} - g_{\nu\sigma}\delta\{^{\sigma}_{\mu\rho}\})\Big]\sqrt{-g}\,\mathrm{d}\Omega, \qquad (19)$$

where the symbols $(\mu\nu)$ denote the symmetrization in the indices μ and ν. The variations $\delta\{^{\sigma}_{\mu\rho}\}$ are expressed in the forms of $\delta g^{\mu\nu}$ and their derivatives. The derivatives are further simplified by partial integrations and we see that the second part of the integrand vanishes. Therefore, we have

$$\int T_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\mathrm{d}\Omega = \frac{\hbar}{2}\int[-i\bar{\psi}\gamma_{(\nu}\psi_{j\mu)} + i\bar{\psi}_{j(\mu}\gamma_{\nu)}\psi]\,\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}\Omega. \qquad (20)$$

Furthermore, the infinitesimal coordinate transformation

$$x^{\mu} = x^{\mu} + y^{\mu}, \qquad (21)$$

where y^{μ} are arbitrary small quantities, when considered in (15), we get[9]

$$\delta\int L\,\mathrm{d}\Omega = -\int T^{\mu}_{\nu j\mu}y^{\nu}\sqrt{-g}\,\mathrm{d}\Omega. \qquad (22)$$

Equating $\delta\int L\,\mathrm{d}\Omega$ to zero, we get

$$T^{\mu}_{\nu j\mu} = 0, \qquad (23)$$

because of the arbitrariness of y^{ν}.

Let us consider the WKB expansions[5] of ψ and $\bar{\psi}$ as

$$\psi = \exp(is/\hbar)\sum_{n=0}^{\infty}(-i\hbar)^{n}a_{n},$$

$$\bar{\psi} = \exp(-is/\hbar)\sum_{n=0}^{\infty}(i\hbar)^{n}\bar{a}_{n}. \qquad (24a,b)$$

Substituting the values of ψ and $\bar{\psi}$ from (24a,b) into (8), (11), (12) and (20) and neglecting the terms which are multiples of $\hbar$ and higher powers of $\hbar$, we get

$$J^{\mu} = \bar{a}_{0}\gamma^{\mu}a_{0}, \qquad (25)$$

$$J^{\mu}_{c} = -\frac{s'^{\mu}}{m}\bar{a}_{0}a_{0}, \qquad (26)$$

$$J^{\mu}_{M} = 0, \qquad (27)$$

and

$$T^{\mu\nu} = \frac{1}{2}[-\bar{a}_{0}\gamma^{\mu}a_{0}s'^{\nu} - \bar{a}_{0}\gamma^{\nu}a_{0}s'^{\mu}]. \qquad (28)$$

In view of (25), (26) and (27), (8) implies

$$\bar{a}_{0}\gamma^{\mu}a_{0} = -\frac{s'^{\mu}}{m}\bar{a}_{0}a_{0}. \qquad (29)$$

This equation simplifies (28) as

$$T^{\mu\nu} = \frac{\bar{a}_{0}a_{0}}{m}s'^{\mu}s'^{\nu}. \qquad (30)$$

Substituting the value of ψ from (24a) into (6) and neglecting the terms which are multiples of $\hbar$ and higher power of $\hbar$, we get

$$(\gamma^{\mu} s_{,\mu} + m) a_0 = 0. \quad (31)$$

For a non-trivial solution of (31) we have the Hamilton-Jacobi equation

$$s'^{\mu} s_{,\mu} = m^2. \quad (32)$$

Therefore, the four-momentum p^{μ} and the four-velocity u^{μ} for the particle are defined as[5]

$$m u^{\mu} = p^{\mu} = -s'^{\mu}. \quad (33)$$

Substituting the value of s'^{μ} from this equation into (30), we get

$$T^{\mu\nu} = (m \bar{a}_0 a_0) u^{\mu} u^{\nu}, \quad (34)$$

which shows that in the classical limit the energy-momentum tensor for the spin-$\frac{1}{2}$ particles behaves like a dust fluid with density ρ_0 given by

$$\rho_0 = m \bar{a}_0 a_0, \quad (35)$$

and hence, in the classical limit the spin-$\frac{1}{2}$ particle follows the geodesic of the space-time geometry[2]

$$u^{\mu}_{;\nu} u^{\nu} = 0, \quad (36)$$

and the momentum density

$$\rho_0 u^{\mu} = m \bar{a}_0 a_0 u^{\mu}, \quad (37)$$

is conserved. In a separate communication we shall discuss the significance of the conservation of the momentum density in the classical limit.

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