FIELD-CONSISTENT FINITE ELEMENTS

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ABSTRACT

Some recent studies into the definition of a new paradigm for the finite element formulation of constrained or nearly constrained multi-strain-field problems of structural analysis are reviewed here.

INTRODUCTION

The finite element method has superceded and virtually replaced all other numerical methods for the digital modelling and solution of complex problems in structural mechanics. It is now a logical component of any automated design-analysis cycle (e.g. CAD/CAM/CAE etc) to assure the structural integrity and efficiency of load carrying members.

At the core of the finite element approach is the idea of replacing continuum regions of the structure by discretizing its elasto-mechanical properties. In software terms, these are done by what are called the mechanics algorithms that carry out this modelling. The accuracy of analysis will depend very much on how faithfully the mechanics algorithms can carry out this modelling process. In this short review, we describe some of the work done at the National Aeronautical Laboratory, Bangalore, India to improve the mechanics algorithms used to replace continuum regions with discretized elements.

EVOLUTION OF THE FIELD-CONSISTENCY PARADIGM

History of the finite element method

The early developments of the finite element method and its initial acceptance proceeded from basic energy and variational principles, backed by simple engineering judgement. Later, the method accepted more rigorous methematical formalizations of its procedures. Variational, weighted residual principles and virtual work theorems offered operational procedures that not only accommodated the physics of the problems but also facilitated the discretization process itself. To some extent, the finite element method is becoming a formal science.

However, the weakest link in the chain remained the discretization strategy—the actual choice of discretization domain (in terms of shape and size and orientation with respect to coordinate axes) and the shape function definitions over that domain that will produce the weighted residual discretized approximations of the continuum behaviour in that domain. Here, the method was practised more in the form of a skilled art rather than as a precise science.

The strategy of discretization followed a broad conceptual scheme which crystallized into a conventional wisdom, the cardinal principles of which were—the piecewise approximations used over element domains should be based on complete polynomials: these should be compatible, i.e. satisfy continuity of these functions or their derivatives, where required, across inter-element boundaries: these functions should be able to represent states of constant strain in the limit; and finally that these functions should be able to recover strain-free rigid body motions.

Introduction of field-consistency concepts

At this stage, the conceptual scheme outlined above still suffered from many inner contradictions. Elements derived from these principles could still behave erratically in important practical applications, although the same elements were in many other respects, excellent. The terms of the paradigms embodied in the conventional wisdom so far did not therefore provide all the sufficient conditions. Conversely it was possible to design elements violating one or more of these conditions which could behave extremely well in the critical applications—an indication that some of these conditions were not even necessary in all instances.

Some of these problems were recorded in the literature as 'shear locking', 'membrane locking', 'parasitic shear' etc. It is believed now that some went unrecorded as there was no adequate framework or terminology with which to classify it. It was not known that these seemingly separate problems were related or that the problem of violently oscillatory stresses within elements originated from this phenomena. Initial rationalisations were made in such terms as the rank, singularity or lack of it, or of the spectral condition numbers of the matrices involved[1]. Other arguments included counting of constraints and free degrees of freedom etc. However, it was soon realized that these heuristic arguments could at best be symptoms of the problem rather than the origin of the actual trouble.

The need for a new paradigm was apparent. The early work of the author and his colleagues[2,3] focussed on some of the 'tricks' used to salvage elements that suffered from shear locking. This was performed within the context of the prevailing paradigms. Its inadequacies, especially with respect to its inability to provide *a priori* estimates of errors became quite obvious. A 'paradigm change' was called for. Subsequent work on curved beams[4], plane stress modelling of flexure[5], plates and shells[6,7] and bricks[8,9] led to the formulation of the new paradigm, called the field-consistency principle, and the definition of new terminology[10,11], a new error norm[12] and its use to bring all these phenomena under a single classification.

A brief overview of the field-consistency approach

The field-consistency approach recognises that problems in structural mechanics (or more generally, in continuum mechanics) which need a description in terms of several strain-fields derived from one or more field variables may require that one or more of these strain fields must vanish in certain constraining limits. Conventional displacement method approaches based on independent low order functions could not ensure that these strain fields would vanish in what, [in our new terminology (see below)], was called a 'consistent' way. These field-inconsistencies led to the enforcement of spurious constraints in the limit.

Next, an operational procedure that could perform an error analysis, keeping track of how field-inconsistencies emerged through the mathematics of discretisation to cause the degraded performance known as 'locking', or poor or delayed convergence even where locking was absent, and violent spurious stress oscillations was devised. This operational procedure, called the functional re-constitution technique was successfully applied to a wide range of problems[2-14]. The conceptual scheme offered new strategies to devise field-consistent elements free of these errors and the field re-distribution strategies[14-17] appear to be the simplest method of designing elements free of locking and spurious stress oscillations.

DEFINITION OF THE FIELD-CONSISTENCY TERMINOLOGY

A short recapitulation of the terminology and vocabulary of the new paradigm follows.

Constrained multi-strain-field problems

Problems which show common characteristics such as locking, poor convergence, stress oscillations etc are first brought under a single classification scheme. These problems need their continuum behaviour to be described by more than one strain-field. In many practical situations, one or more of these strain-fields must be constrained. The discretized functionals for the constrained part of the strain energy of a finite element should be able to vanish in a way that ensures that only true constraints emerge in the limit. Where it fails to do so, we have the

Table 1 *Some constrained multi-strain-field problems*

Name of exercise	Strain-fields: Unconstrained	Strain-fields: Constrained
Plane stress, Plane strain, 2D; 3D Brick Modes of flexure	Normal	Shear
Plane strain, 2D; 3D Bricks modes of nearly incompressible elasticity	Distortional	Dilatational
Shear flexible Beams (Timoshenko) and Plates (Mindlin)	Bending	Transverse shear
Inextensional bending of curved beams; thin shells	Bending	Membrane
Shear flexible thick curved beams; shells	Bending	Membrane and shear

problems associated with field-inconsistency. Table 1 gives a representative list of exercises which come under this class.

Field-consistency

The critical step in the discretization process is the selection of interpolation functions to describe the field variables over the element domain. From these functions, one can compute the strain fields also as interpolations associated with the constants of the field functions by obtaining the correct derivatives of the field variables. In a multi-strain-field problem, these strain fields will have as coefficients, terms from more than one field variable. Depending on the order of derivatives of each field variable appearing in the definition of that strain field and on the order of the interpolation functions used for each contributing field variable, the coefficients of the strain field interpolations may have constants from all contributing field variable interpolations or from only one or some of these. In some limiting cases of physical behaviour, it will become necessary for these strain fields to be constrained to be zero values.

Where the discretized strain-field is such that all the terms in it (i.e. the constant, linear, quadratic etc) have, associated with it, coefficients containing contributions from all the independent interpolations of the field variables that appear in the definition of that strain-field, the constraint that appears in the limit can be correctly enforced. Such a representation is said to be field consistent. The constraints thus enforced are called 'true constraints'. Where the discretized strain field has coefficients in which the contributions from some of the field variables are absent, the constraints may incorrectly constrain the contributions from the field variables present. These are called 'spurious constraints'.

In this review, we shall report the results from an examination of one of these exercises to show that field-inconsistency leads to the very poor performance reflected as locking, poor convergence, stress oscillations etc.

AN ILLUSTRATIVE EXAMPLE: THE POOR BENDING RESPONSE OF A SHEAR FLEXIBLE BEAN ELEMENT

This is the simplest and earliest example in which the problems of field-inconsistency were identified and modelled mathematically through definitions and operations of the new paradigm[2]. Here, we shall cite very briefly, the results from that study to illustrate the concepts introduced above.

Historical introduction

A two-noded beam element based on the shear flexible Timoshenko beam theory—two degrees of freedom at each node follow from a transverse displacement w and a section rotation θ which are independent field variables—needs only C^0 continuity and can be based on simple linear interpolations. Early experiments with this element proved to be disastrous. No record of such an element appeared until 1977 when the deceptively elementary 'trick' of using a reduced integration of the shear strain energy, produced a remarkably accurate element[18]. Initially, this was dismissed as a 'useful trick', implying that reduced integration introduced errors that compensated somehow for the other constraining errors.

A reworking of the same shear flexible linear beam element[2] gave a better understanding of the principles involved. The improvement brought about by reduced integration was rationalized in terms of the types of constraints seen at the penalty limits. The foundation of the functional re-constitution procedure was laid in this paper and it was seen to give very accurate predictions of how an exactly integrated element would behave in static and dynamic problems. The elaboration in terms of field-consistency interpretations came later[4-17]. In this section, we summarise the salient findings of our earlier study[2].

The FI and FC elements

The conventional element (FI for field-inconsistent) uses linear functions for the field-variables w and θ,

$$w = a_0 + a_1 \ (x/l), \quad (1a)$$

$$\theta = b_0 + b_1 \ (x/l). \quad (1b)$$

These lead to the following interpolation for the shear strain-field

$$\gamma = (b_0 - a_1/l) + b_1 \ (x/l). \quad (2)$$

For a thin beam, the shear strains must vanish. The discretized constraints that emerge are,

$$b_0 - a_1/l \rightarrow 0, \quad (3a)$$

$$b_1 \rightarrow 0. \quad (3b)$$

In our terminology, constraint (3a) is field-consistent as it contains constants from both the contributing field-variables relevant to the shear strain field. These constraints can then accommodate the true Kirchhoff constraint in a physically meaningful way. In direct contrast, constraint (3b) contains only a term from θ. A constraint imposed on this will lead to an undesired restriction of θ—this is the 'spurious constraint' that leads to 'shear locking' and violent disturbances in the shear force prediction over the element, as we shall see presently.

A field-consistent element (henceforth FC element) is obtained by evaluating the shear strain energy so that only the consistent term will contribute to the shear strain energy. This will show a remarkably improved behaviour and will be free of shear force oscillations.

Figure 1 shows a typical illustration of what we mean by the very poor behaviour of the FI element as compared to the dramatic improvement in efficiency obtained by making it field-consistent (FC). A map of percentage error E can be made for N-element models of a cantilever beam under tip shear load for various ratios of length of beam L to beam depth t. Figure 2 shows this relationship on logarithmic scales. It is evident that over the practical range in which Timoshenko beam theory is appropriate (i.e. say $L/t = 5$ to $L/t = 1000$), the FI models are virtually impractical to use—needing as many as $100(L/t)$ elements to achieve the same accuracy as obtained with 10 FC elements!

It is possible to derive an accurate prediction for the manner and degree of error due to shear locking in the FI element through a technique called the functional re-constitution procedure. We omit the details now but show how such an

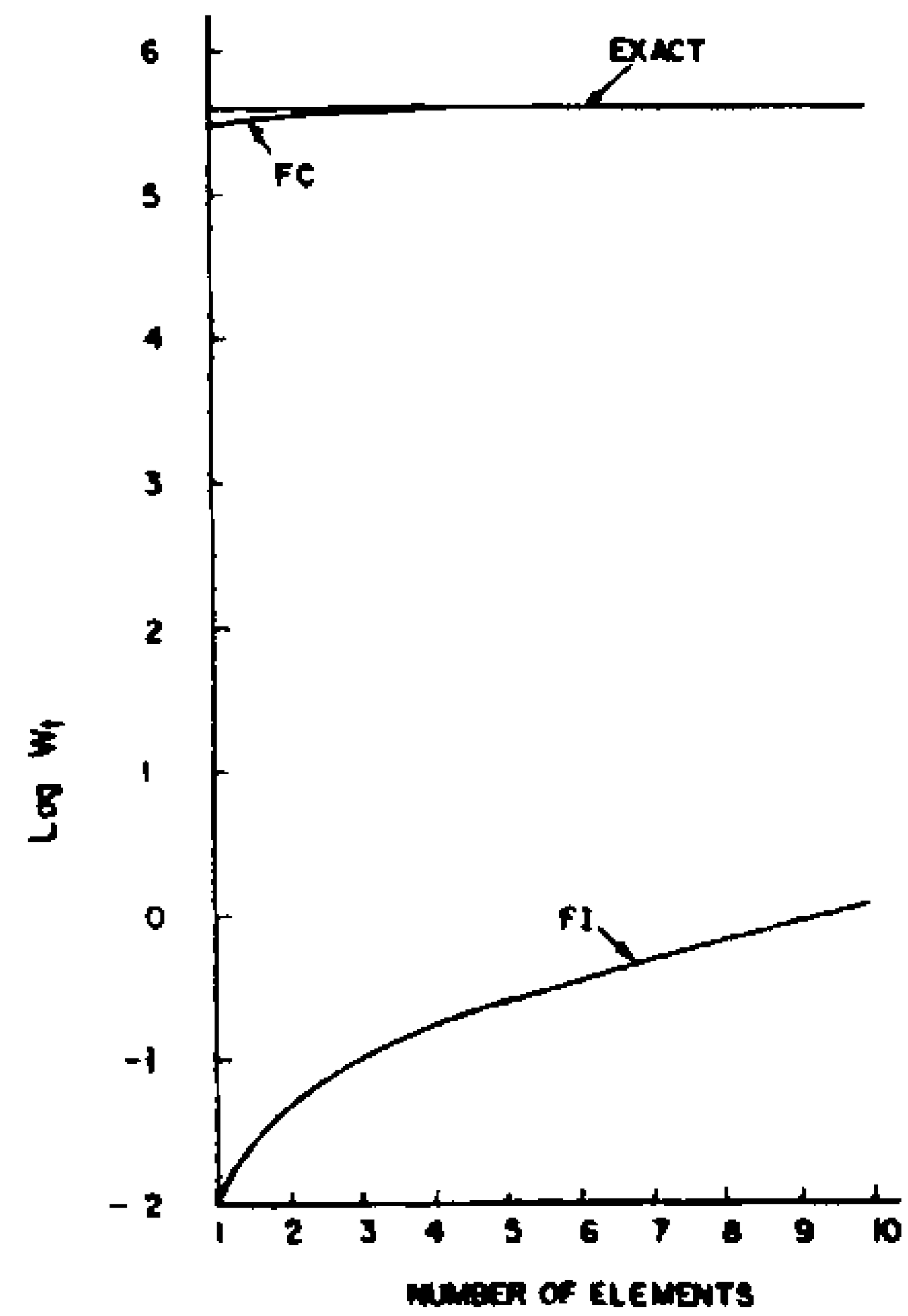

Figure 1. Convergence plot for tip deflection of a cantilever beam under tip shear load for $L/T = 10^4$.

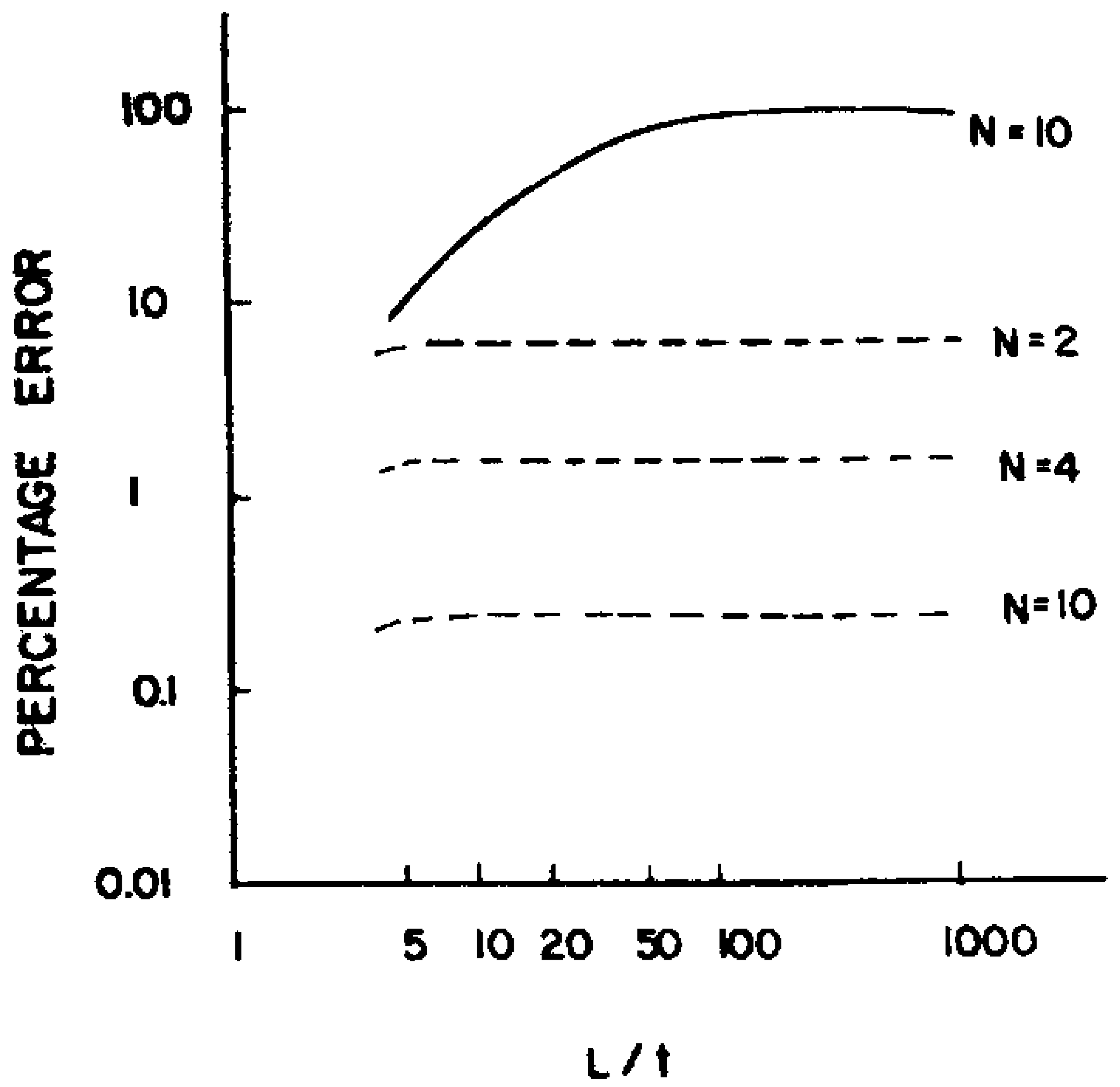

Figure 2. Error map for N-element model of Timoshenko cantilever beam with tip shear load.—— F1 modelling. ---FC modelling.

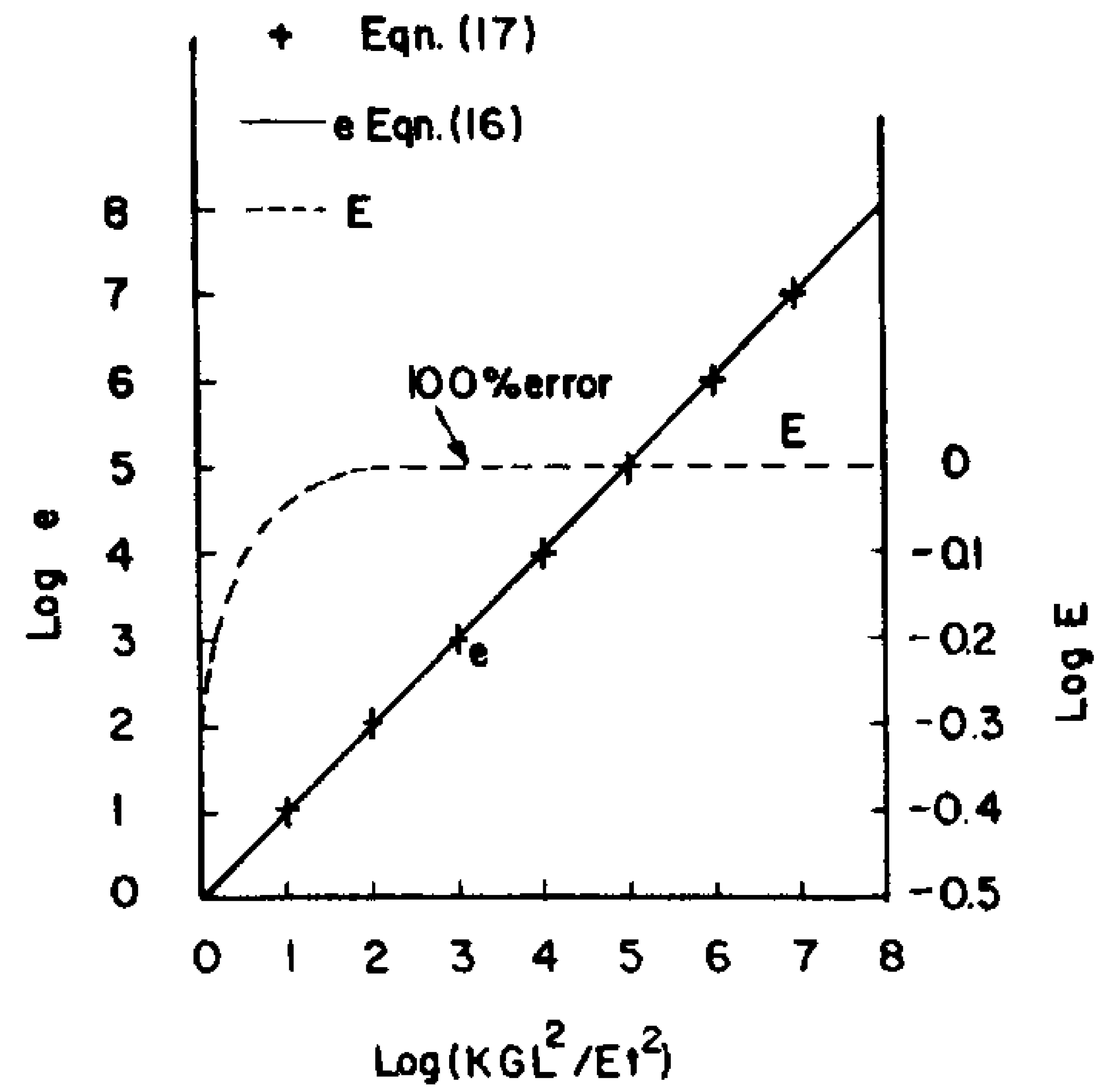

Figure 3. Error norms e, E as function of penalty multiplier for cantilever beam under tip shear force.

apriori prediction can be confirmed in a numerical example. Exact solutions are available for the static deflection w of a Timoshenko cantilever beam of length L and thickness t under a vertical tip load. If w(fem) is the result from a finite-element digital computation using elements of length 21, and thickness t, the additional stiffening parameter for such a model is

$$e(\text{fem}) = w/w(\text{fem}) - 1. \qquad (4)$$

From our studies[2], we can postulate the error model prediction to be

$$e = KGAL^2/3\text{EI}. \qquad (5)$$

Figure 3 shows the variation of e with the structural parameter that denotes the penalty multiplier in this case, namely KGL^2/Et^2 for the case presented in figure 1. The crosses indicate the additional stiffening parameter computed for the finite element computation (equation (4)) and the solid line shows the variation as predicted by the error model(equation (5)).

With this example, we were able to demonstrate that the field consistency approach together with the functional reconstitution procedure could give a powerful 'falsifiability' capability i.e. we are able to predict that an element, designed in a certain way and containing certain measurable deficiencies should behave in a precisely quantifiable erratic way.

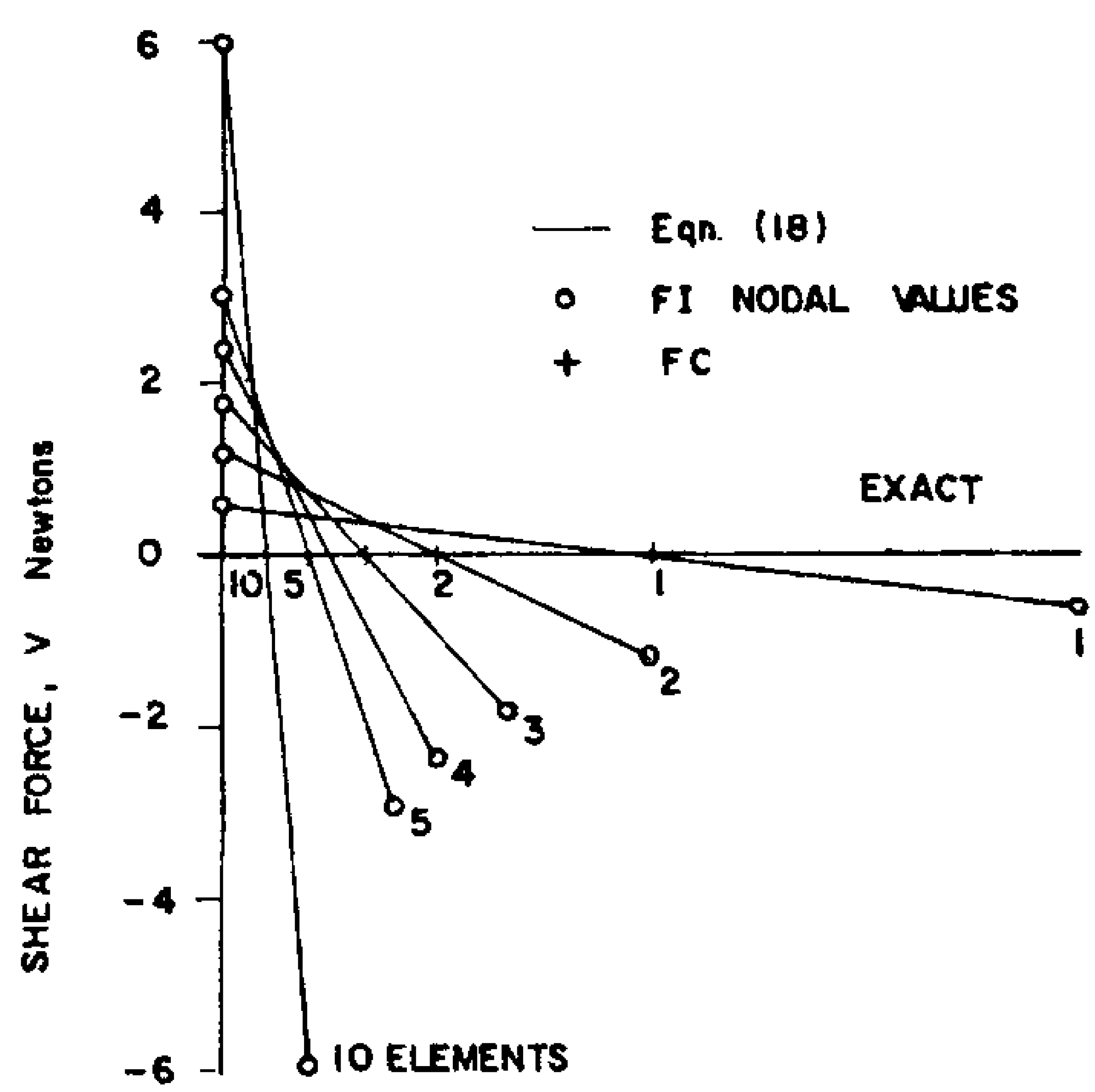

Figure 4. Shear force oscillations in cantilever beam with tip moment.

An extension of the functional re-constitution procedure, allows us to show that the inconsistent constraint will further appear as violent spurious shear force oscillations. Figure 4 shows the shear force oscillations in a typical problem—a straight cantilever beam with a concentrated moment at the tip. One to ten equal length field-inconsistent elements were used and shear forces were computed at the nodes of each element. In each case, only the variation of the element at the fixed end is shown, as the pattern repeats itself over all other elements. At element mid-nodes, the actual shear force, i.e. $V = 0$ is correctly reproduced. Over the length of the element, the oscillations are seen to be linear functions corresponding to the inconsistent term. Also indicated by the solid lines, is the prediction made by the functional re-constitution exercise.

CONCLUDING REMARKS

In this review, we have surveyed contributions to an area that has generated considerable interest and controversy in recent years. We understand that this uncertainty resulted from the incompleteness of the current paradigms that informed the state of the art in the finite element method. It was also clear that a re-definition of paradigms was needed.

The first task was to classify the problem area and to give this class a name for quick reference of the principles and issues involved—it is hoped that the requirement of 'field-consistency' will join the other more well-known principles used in constructing finite elements.

Another important task was to provide it auxiliary terms of reference and procedures that can give it an error-analyses capability. The concept of 'errors of the second kind' to delineate a special form of discretisation errors and the 'additional stiffening parameter' were found necessary to make *a priori* error models and *a posteriori* evaluations of elements used in this class of problems.

The extension of these concepts to other similar field problems and the invention of new techniques or 'tricks' that can restore field-consistency will be interesting exercises. Of more immediate practical relevance will be the re-design of the existing element libraries of major general purpose software packages so that many capabilities which did not exist can now be included.

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ANNOUNCEMENTS

SEMINAR ON PROJECT WORK IN CHEMISTRY

The Department of Chemistry, Loyola College, Madras, will be organizing a two day seminar on 'Project work in Chemistry' during **26th and 27th of July 1986**. The seminar will highlight the mechanics and conduct of project work to make chemical education creative, stimulating and productive and will discuss the importance and impact of project work in solving social and industrial problems through relevant projects.

Intending participants are requested to submit their abstracts in 250 words to the 'Convener, Seminar on Project work in Chemistry, C/o Head of the Department of Chemistry, Loyola College, Madras 600 034.'

SOCIETY FOR BIOMATERIALS AND ARTIFICIAL ORGANS-INDIA

(Regd. No. 110/86)

The Society for Biomaterials and Artificial Organs-India has been formed with its registered office at Division of Bio-surface Technology, BMT Wing, Sree Chitra Tirunal Institute for Medical Sciences and Technology, Poojapura, Trivandrum 695 012, India.

Objective: Biomaterials and their applications as artificial organs is an emerging area for biotechnologists, chemists, physicists, biologists, engineers and medical professionals. However, the fundamental and applied research in this multidisciplinary field is still in its incipient stages in India. The objective of this society will be not only to cultivate an atmosphere nationwide for the growth of basic and technological oriented research in biomaterials and artificial organs but also to bring the professionals from various faculties together to improve the quality of their approach in future research. This will be done by communication to all members, by developing a society's Newsletter and also by arranging workshop with laboratory demonstrations, where learning and exchange of relevant concepts can be stimulated every year.

Management: The management of the affairs of the society shall be entrusted in accordance with rules and regulations of the society to the executive council.

Membership of the society shall be open to all individuals interested in the subject. For further information, please contact, Dr Chandra P. Sharma, Division of Biosurface Technology, BMT wing, SCTIMST, Poojapura, Trivandrum 695 012, India.