MAGNETO-VISCOELASTIC INTERACTION ON ONE-DIMENSIONAL WAVE PROPAGATION

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ABSTRACT

Analytical results are reported for magnetic viscoelastic interaction on one-dimensional wave propagation in a solid medium. The relevent frequency equations have been obtained by solving equations of viscoelasticity of Reiss type, taking into account the effect of a magnetic field and the electromagnetic equations of Maxwell. The nature of frequency has been discussed in different cases.

1. INTRODUCTION

The study of interaction of an externally applied magnetic field on the elastic motion and deformation of a solid is known as magnetoelasticity. This phenomenon is interlocking in character and has extensive practical applications in diverse fields such as geophysics, acoustics, damping of acoustic waves in a magnetic field and so on. In recent years the problems of magneto elastic waves and vibrations are receiving greater attention by many investigators[1-5]. Although, the effect of magnetic field on the elastic field is small, the theory of magnetoelasticity is developed owing to its importance in geophysical, seismological and cosmological problems and to some extent to practical problems when the magnetic field is sufficiently large[6-8]. In discussing the propagation of seismic waves from the earth's mantle to its core, Cagniard[9] suggested that the existence of the earth's magnetic field should be considered for explaining certain phenomena of magnetoelastic waves. Subsequently, Knopoff[10] investigated the effects of magnetic field on the propagation of elastic waves on a geophysical scale. Magnetoelastic interaction on geophysical problems were investigated by other workers[11,12]. Datta[13] considered the problem of magnetoviscoelastic interaction on radial vibration of a cylinder. The problem of magnetoelastic wave propagation in a thermal field was investigated by Paria[14]. Some problems of magnetoelastic, magnetoviscoelastic waves and vibrations were investigated by Datta[15,16]. As a sequel to these papers the present paper is an attempt to investigate some aspects of dispersion of waves in viscoelastic solid of Reiss type acted upon by a magnetic field. The relevant frequency equations have been obtained and the nature of frequency has been discussed in different cases.

2. GENERAL THEORY AND FUNDAMENTAL EQUATIONS

To investigate the interaction of the magnetic field and the viscoelastic field, we need to solve the appropriate equations describing the above two fields. The Maxwell equations governing the electromagnetic field are

$$\text{Curl}\,\mathbf{H} = \mathbf{J}, \quad \text{div}\,\mathbf{B} = 0$$
$$\text{Curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{B} = \mu_e \mathbf{H} \tag{2.1}$$

where the displacement current is neglected. The generalised Ohm's law in the deformable medium is

$$\mathbf{J} = \sigma\left[\mathbf{E} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B}\right] \tag{2.2}$$

In eqs (2.1) and (2.2), **H**, **B**, **E** and **J** respectively denote the magnetic intensity, magnetic induction, electric intensity and current density vectors. μ_e is the magnetic permeability of the body **u** represents the displacement vector in the strained solid and σ is the electrical conductivity.

The stress-strain relation in a viscoelastic medium of Reiss type[17] may be taken as

$$\tau_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} + \lambda' \frac{\partial \theta}{\partial t}\delta_{ij} + 2\mu' \frac{\partial e_{ij}}{\partial t} + \lambda'' \frac{\partial^2 \theta}{\partial t^2}\delta_{ij} + 2\mu'' \frac{\partial^2 e_{ij}}{\partial t^2} \tag{2.3}$$

where τ_{ij} is the stress tensor, e_{ij} is the strain tensor, λ, λ', λ'', μ, μ', μ'' being the material constants.

The strain displacement relations are

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.4}$$

The stress equations of motion are

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i, \tag{2.5}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector ρ the density and $\mathbf{F} = (F_1, F_2, F_3)$ is the body force per unit volume.

If there are no body force apart from the Lorentz force, we take

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{2.6}$$

From (2.1) to (2.6) it follows that the electromagnetic field is in interaction with the viscoelastic field.

3. REDUCTION TO THE CASE OF A UNI DIRECTIONAL MOTION

We consider the case of the propagation of disturbance in one direction, say x-direction only. The z axis is taken in the direction of original magnetic field (primary) H_z. Then the displacement $\mathbf{u}$ has the components $(u, 0, 0)$ where $u(x, t)$ and all the above vector quantities depend on x and t only. Also, let the magnetic field be such that

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \tag{3.1}$$

where $\mathbf{H}_0 = (0, 0, H_z)$ is the initial magnetic field acting parallel to z-axis and $\mathbf{h} = (0, 0, h_z)$ is a small perturbation in the field.

From the first equation of (2.1) and (2.2) we have

$$\sigma E_y = \frac{H_z}{\nu_H}\frac{\partial u}{\partial t} - \frac{\partial h_z}{\partial x}, \tag{3.2}$$

where $\nu_H = 1/\mu_e \sigma$. Also, we have from the third equation of (2.1) and (2.2)

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_z \frac{\partial^2 u}{\partial x \partial t}. \tag{3.3}$$

The first and last equations of (2.1) leads to

$$\mathbf{J} \times \mathbf{B} = \left[-\mu_e H_z \frac{\partial h_z}{\partial x}, 0, 0 \right] \tag{3.4}$$

where we have neglected the products of small quantities of $\mathbf{u}$ and $\mathbf{h}$. Eliminating e_{ij} from (2.3) and (2.4) and substituting the resulting components of stresses in the equations of motion (2.5) and using (2.6) and (3.4), we get

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda' + 2\mu')\frac{\partial^3 u}{\partial x^2 \partial t} + (\lambda'' + 2\mu'')\frac{\partial^4 u}{\partial x^2 \partial t^2} - \mu_e H_z \frac{\partial h_z}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{3.5}$$

4. SOLUTION OF THE PROBLEM AND DISCUSSIONS OF THE RESULTS

We consider the case when electrical conductivity of the medium is finite. In this case, we take

$$\begin{aligned} u &= u' \exp[i(\gamma x - \omega t)] \\ h_z &= h' \exp[i(\gamma x - \omega t)] \\ E_y &= E' \exp[i(\gamma x - \omega t)] \end{aligned} \tag{4.1}$$

where u', h', E' are constants, γ is the wave velocity and ω is the frequency of the wave.

Substituting the values of u, h_z, E_y in (3.2), (3.3) and (3.5) and then eliminating u', h', E' from resulting equations, we have the wave velocity equation given by

$$\rho(i\omega - \nu_H \gamma^2)\{\omega^2(1 + C_1\gamma^2) - A_1\gamma^2 + iB_1\gamma^2\omega\} - \mu_e H_z^2 \gamma^2 \omega^2 i = 0 \tag{4.2}$$

where,

$$A_1 = \frac{\lambda + 2\mu}{\rho}, \quad B_1 = \frac{\lambda' + 2\mu'}{\rho}, \quad C_1 = \frac{\lambda'' + 2\mu''}{\rho} \tag{4.3}$$

Equation (4.2) can be written in the form

$$i\omega^3 + P\omega^2 + \varphi i\omega - R = 0 \tag{4.4}$$

where,

$$P = \frac{\nu_H \gamma^2(1 + C_1\gamma^2) + B_1\gamma^2}{(1 + C_1\gamma^2)}$$

$$\varphi = \frac{A_1\gamma^2 + B_1\gamma^4\nu_H + \dfrac{\mu_e H_z^2 \gamma^2}{\rho}}{(1 + C_1\gamma^2)} \tag{4.5}$$

$$R = A_1 \nu_H \gamma^4$$

We will study two different cases: (1) when the wave length $L = 2\pi/\gamma$ is real, *i.e.* γ is real, (2) when the frequency ω is real.

Case 1. If we take γ to be real, then from the coefficients of (4.4) it follows that the possibilities for the roots of ω are (i) 3 imaginary, no real (ii) one imaginary, 2 real. For an imaginary root of ω say $(\alpha + i\beta)$ any equation of (4.1) is of the form $C\exp[i\{\gamma x - (\alpha + i\beta)t\}] = C\exp[i(\gamma x - \alpha t)\exp(\beta t)]$. This sort of solution admits of a variety of interpretations depending on the nature of α and β. Since we are considering viscoelastic waves, they can reflect damping characteristics in time, only when $\beta < 0$. Also, it is evident from (4.4) and (4.5) that the frequencies of the magnetoviscoelastic waves depend on the wavelength in a nonlinear manner and so different wavelengths will propagate with different phase velocities. We can, therefore, conclude that magnetoviscoelastic waves are dispersive.

Case 2. If we take ω to be real we rewrite (4.2) in the form

$$\gamma^4 - S\gamma^2 + T = 0 \tag{4.6}$$

where,

$$S = \frac{\omega^2 v_H - B_1\omega^2 + i\omega^3 C_1 + i\omega A_1 + \dfrac{i\omega\mu_e H_z^2}{\rho}}{A_1 v_H - B_1 v_H i\omega - C_1 v_H \omega^2}$$

$$T = \frac{i\omega^3}{B_1 v_H i\omega + C_1 v_H \omega^2 - A_1 v_H} \tag{4.7}$$

We find that γ has two pairs of equal and opposite imaginary roots. Let the roots be $\pm(\alpha' + i\beta')$, $\pm(\gamma' + i\delta')$. When the roots are $(\alpha' + i\beta')$ and $(\gamma' + i\delta')$ then the amplitude of the wave contains the term like $\exp(-\beta' x)$ and $\exp(-\delta' x)$ which show that the amplitude die out in the x-direction, provided β' and δ' are positive.

Group Velocity: As already remarked, the waves are dispersive and as the group velocity of wave is a characteristic velocity that represents the speed with which energy is propagated. We now calculate the group velocity of the dispersive waves.

Putting $C = \omega/\gamma$ (phase velocity) in (4.2), we have,

$$C^4 + P_1 C^2 + \varphi_1 = 0 \tag{4.8}$$

where,

$$P_1 = \frac{\{(iB_1\omega - A_1) - v_H i\omega(1 + C_1\gamma^2)\}}{(1 + C_1\gamma^2)}$$

$$\varphi_1 = \frac{\mu_e H_z^2}{\rho(1 + C_1\gamma^2)} \tag{4.9}$$

Now, group velocity C_g of the wave can be written as

$$C_g = \frac{C}{\left(1 - \dfrac{\omega}{C}\dfrac{\partial C}{\partial \omega}\right)}$$

$$= \frac{C(2P_1C^2 + 4C^4)}{(2P_1C^2 + 4C^4) + \omega C^2 \dfrac{\partial P_1}{\partial \omega} + \omega \dfrac{\partial Q_1}{\partial \omega}} \tag{4.10}$$

[by (4.8) and (4.9)]

It is seen that if the frequency of the wave increases *i.e.* for large values of ω, $C_g \to \infty$. This implies that a wave packet consisting of infinitesimally short wavelengths will propagate with infinte velocity which is physically impossible. Therefore, infinite C_g with $\omega \to \infty$ is inadmissible in our problem, however, for small values of ω, we find that the group velocity of the waves is equal to the phase velocity of the wave.

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