RECENT DEVELOPMENTS IN MATRIX ENSEMBLE THEORY

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ABSTRACT

Various phases of development of compound-nucleus reactions and some recent applications of the random matrix theory in other areas of physics are described. The new technique of using a resolvent and Grassmann variables in the study of random matrix ensembles is illustrated using a two-dimensional Gaussian orthogonal ensemble.

INTRODUCTION

One of the most interesting and challenging areas in manybody physics has been the study of nuclear reactions which pass through the formation of a compound nucleus. In the early days these reactions were studied by bombarding a heavy nucleus like ^{238}U with a beam of monoenergetic slow neutrons[1]. A very large number of resonances were observed in the total cross section. It was soon realized that these resonances were due to the decay of the quasi-stationary states of the compound nucleus ^{239}U. The compound nucleus picture was put forward by Niels Bohr and many of the consequences were later verified experimentally.

The most difficult aspect of these reactions was to develop a theory to explain the observed behaviour of the widths and positions of these resonances. The difficulty here mainly arose because the compound nucleus states lie above the nuetron binding energy and therefore involve many nuclear configurations. It was this complexity which led Wigner[2] to introduce the idea of a matrix ensemble as an alternative to the usual diagonalization problem of manybody physics.

Further developments in this field took place when overlapping compound nucleus resonances were observed where the effects of unitarity of the low energy scattering matrix became important[3]. The theoretical problem became even more complex as one had to find ways of calculating the averages of various cross sections[4].

Recently the matrix ensemble theory has found new applications in the study of collective modes[5] and the conductivity of electrons in a random potential[6]. In these studies new techniques have been developed to study the ensemble averages of the Hamiltonian matrix elements. In the present paper we would first like to present a few of the earlier theoretical results and then describe briefly some of the new results obtained using Grassmann integration[7].

FORMULATION

For the physical systems the relevant matrix ensemble is Gaussian orthogonal ensemble[8] (GOE).

We consider a real-symmetric $N \times N$ Hamiltonian matrix, then the joint distribution of its N diagonal and $\frac{1}{2}N(N-1)$ off-diagonal matrix elements can be written as

$$P(H) = K \exp(-\mathrm{Tr}\, H^2), \tag{1}$$

where K is the normalization constant. One of the earliest results is the joint eigenvalue distribution $P(E_1, \ldots, E_N)$, obtained by integrating out the eigenvectors, it is given by[8]

$$P(E_1, \ldots, E_N) = K \exp(-\sum E_i^2) \prod_{\mu<\nu} |(E_\mu - E_\nu)|. \tag{2}$$

This is known as Wishart distribution.

All the theoretical results for the well spaced compound nucleus resonances like Wigner's spacing distribution[8] are derived from expression (2).

In the study of the average cross-sections one starts from the scattering matrix

$$S = S_0\left[1 - i\sum_\mu \frac{g_\mu \times g_\mu}{E - Z_\mu}\right], \tag{3}$$

where g_μ are the amplitudes and Z_μ are the poles of the S-matrix. The properties of g_μ and Z_μ can be studied[3] by using the relations which connect them with the eigenvalues and eigenvectors of H, the distribution of which is given by expression (1). A good example is provided by the average value of S for the purely elastic scattering case. This is given by

$$\langle S \rangle = \exp-(\pi \langle \Gamma_\mu \rangle / D), \tag{4}$$

where S^0 is taken to be unity and $\langle \Gamma_\mu \rangle$, D are the average width and spacing.

In the recent studies one introduces the resolvent

$G(Z)$ given by

$$G(Z)=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{Z-E_i}, \quad (5)$$

to study various ensemble averaged quantities. In terms of the Hamiltonian it can be written as

$$G(Z)=\frac{1}{N}\left.\frac{\frac{\partial}{\partial\xi}\det(\xi-H)}{\det(Z-H)}\right|_{\xi=Z}. \quad (6)$$

Thus the advantage of the resolvent is that one can directly use the distribution of H to study its ensemble averages. It is here that one makes use of Grassmann integration to represent the determinant by an integral over Grassmann variables[7].

As a simple example if we consider a two dimensional case then it can easily be shown using expressions (1) and (6) that the ensemble average $g(z)$ of the resolvent $G(Z)$ is given by

$$g(Z)=\frac{2}{i}\int_0^{\infty}\exp(-t^2+2itZ)M\left(-\frac{1}{2},\frac{1}{2},\frac{t^2}{2}\right)dt, \quad (7)$$

where $M(a,b,X)$ is the confluent hypergeometric function[9].

From expression (7) one can derive all the ensemble-averaged quantities involving single eigenvalue, *e.g.* the probability density function of the single eigenvalue $P(E)$ is given by

$$P(E)=\frac{2}{\pi}\int_0^{\infty}dt\exp(-t^2)\cos(2tE)M\left(-\frac{1}{2},\frac{1}{2},\frac{t^2}{2}\right), \quad (8)$$

which on carrying out integrations[10] over t gives

$$P(E)=\frac{1}{\sqrt{2\pi}}\exp(-E^2)M\left(-\frac{1}{2},\frac{1}{2},-E^2\right). \quad (9)$$

When N becomes large one uses the saddle point method[11] to evaluate $g(z)$. In this way one has used the new technique to evaluate products of scattering matrix elements of S non-perturbatively[12] and derived an exact expression for the average compound-nucleus cross section which is valid for all transmission coefficients. Since the final expression is fairly long, it is not given here but can be found in ref. 12.

CONCLUSIONS

We have described various phases of developments in the area of compound nucleus reactions. The theoretical concepts are now finding new applications in other areas of physics. The new theoretical techniques which make use of resolvent and Grassmann variables are described and illustrated using a two-dimensional case.

It is interesting to note that new approximations can be obtained using expressions of the form given by (7), *e.g.* if we retain the first two terms in the expansion of confluent hyper geometric function in expression (7), we get the following approximate single eigenvalue probability density function

$$P_{\mathrm{ap}}(E)=\frac{3}{4\sqrt{\pi}}[\exp(-x^2)](1+\frac{2}{3}x^2). \quad (10)$$

This turns out to be a very good approximation to the exact distribution given by (9). The new technique has also been used recently to derive an exact expression for the average compound-nucleus cross-section[12] which is valid for all transmission coefficients.

25 April 1985

1. Mahaux, C. and Weidenmüller, H. A., *Shell model approach to nuclear reactions*, North-Holland, Amsterdam, 1969.
2. Wigner, E. P., *Can. Math. Cong. Proc.*, University of Toronto Press, Toronto, Canada, 1957, p. 174.
3. Sandhya Devi, K. R. and Ullah, N., *J. Math. Phys.*, 1972, **13**, 316.
4. Ericson, T., *Ann. Phys. (N.Y.)* 1963, **23**, 390.
5. Edwards, S. F. and Jones, R. C., *J. Phys. A*, 1976, **9**, 1595.
6. Efetov, K. B., *Sov. Phys. JETP*, 1982, **55**, 514.
7. Balian, R. and Zinn-Justin, J. (Ed.), *Methods in field theory*, Les Houches Ecole d'Etede Physique Theorique, 1975, Session XXVIII, p. 1, North-Holland, Amsterdam, 1976.
8. Mehta, M. L., *Random matrices*, Academic Press, N. Y., 1967.
9. Abramowitz, M. and Stegun, A., *Handbook of mathematical functions*, DoVer, N. Y., 1965.
10. Gradshteyn, I. S. and Ryzhik, I. M., *Tables of integrals, series and products*, Academic Press, N. Y., 1965.
11. Verbaarschot, J. J. M., Weidenmüller, H. A. and Zirnbauer, M. R., *Phys. Rev. Lett.*, 1984, **52**, 1597.
12. Verbaarschot, J. J. M, Weidenmüller, H. A. and Zirnbauer, M. R., *Grassman integration and the theory of compound-nucleus reactions*, MPI preprint V20, 1984.