THE EVOLUTIONARY SEQUENCE OF FIVE GEOMETRIC SOLIDS

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ABSTRACT

This study showed that the so-called five regular solids, namely; tetrahedron, octahedron. hexahedron, icosahedron and dodecahedron can be separated into two groups based on the degree of their regularity or symmetry. The octahedron and the icosahedron constitute the less regular of the five solids. The octahedron can be inscribed in a regular hexahedron with an edge length of about one and a half times longer than that of a regular octahedron. Similarly, an icosahedron can be inscribed in a regular dodecahedron in which the edge length of the pentagonal planes equals the edge length of the triangular planes of an icosahedron. The tetrahedron, (4-sided), the hexahedron (6-sided) and the dodecahedron (12-sided) are on the other hand the truly regular solids and their structural complexity follows in that order.

In the two less regular of the five solids, 'false' planes exist while 'true' planes are absent. The reverse is the case for the remaining three solids that are more regular. This paper discusses a probable evolutionary sequence of these solid geometries.

INTRODUCTION

Interest in the geometry of solids and its applications has increased in recent years in areas of crystallography, solid state physics, graphics, topology, quantum physics, virology, architecture, and in theoretical models on ecological, biological and chemical systems among others.

Geometric paterns were often presented in two rather than three dimensions in the manuscripts of ancient religions such as Buddhism[1], and Islamic sufism[2,3]. Records or geometric solids dated to the works of Plato around 4th century BC, although there are strong indications that the Pythogoreans had an earlier contact with these solids[4]. Kepler[4] and Euler[5] among others were also interested in the geometric regularity of the tetrahedron, octahedron, hexahedron, dodecahedron, and icosahedron which ancient Greek philosophers regarded as cosmic figures.

Euler[5] way back in the 18th century related the number of faces, f, of edges, e and of vertices, v, for any given regular convex polyhedron with the equations

$$f+v=e+2, \tag{1}$$

and

$$nf=mv=2e, \tag{2}$$

which satisfy a regular polyhedron with n sides and m edges arising from one vertex. Five solutions to these questions yielded the five cosmic or platonic solids and no more as shown[4] in table 1.

Table 1 *Geometric properties of the five regular solids (after Pedoe[4])*

Solid	f	v	e	m	n	Face (Plane)
Tetrahedron	4	4	6	3	3	Triangle
Hexahedron	6	8	12	3	4	Square
Octahedron	8	6	12	4	3	Triangle
Dodecahedron	12	20	30	3	5	Pentagon
Icosahedron	20	12	30	5	3	Triangle

f; number of faces, e; number of edges, v; number of vertices, n; number of sides arising from a vertex, m; number of edges arising from a vertex.

In recent times, references to the so-called five regular solids have increased. In previous reports by Pedoe[4], and Takac's,[6] among others, these solids were arranged from the one with the least number of sides (tetrahedron) to the one with the highest number of sides, (icosahedron). This sequence does not, however, reflect the evolutionary pattern of these solids. There is also a descrepancy in the blanket description of these solids as regular with a disregard to the differences in their degree of symmetry and the interrelationships between and among the solids. The establishment of the evolutionary sequence of these solids which is not apparent in earlier studies, is the objective of this paper. This attempt will probably increase the appreciation of those who have one type of interest or the other in these solid geometries.

Although the two-dimensional (2-D) models employed in the illustration of biological systems were quite effective[7,8], the three-dimensional (3-D) models will probably still better illustrate ideal interactions within biological or agro-ecological systems. The same is even more true in the case of rock minerals where the mineral particles are patterned in three dimensions. However, most of the manifestations in nature deviate from ideal patterns of ordered creation which these five solids describe. A struggle towards the realization of this ideal appears to be the riddle of evolutionary progressions with the attendant trials and errors that can be seen in nature.

OBSERVATIONS

Two-dimensional geometric forms had been used to illustrate interaction patterns between and among associated crops in intercropping systems[7] and also among maize plant parts and growth factors[8,9].

Efforts were subsequently made between 1980 and 1983 to model these 2-D patterns in 3-D using cardboard models. These efforts yielded amongst others, the following solid geometries; tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron. The author was not aware of previous reports on these geometric solids prior to the modelling. The appreciation for these solids has increased following an awareness of their re-occurrence in different disciplines.

The experience gained during the process of modelling the shapes is here used to arrange these solids according to the logical sequence in which one shape is transformed to other shapes. Discussions about the evolution of these solid geometries in this work focus on ideal interaction patterns which *albeit* are rare in nature.

(a) A tetrahedron

(b) Three tetrahedra with an octahedron inscribed within.

(c) Twin tetrahedra with eight vertices and six potential faces.

(d) A hexahedron (cube) with eight vertices and six square faces.

Figure 1. Evolutionary stages of the hexahedron from a tetrahedron.

DISCUSSION

The probable sequence of evolution of five solid geometric forms—tetrahedron, octahedron, hexahedron, icosahedron and dodecahedron

A tetrahedron which has four triangular faces (figure 1a) is the simplest 3-D space enclosure. The surface-to-volume ratio is used to describe the efficiency of a 3-D structure in the enclosure of space. Among all possible space enclosures systems, the tetrahedron has the smallest surface-to-volume ratio while the sphere has the highest. Other geometric forms fall between these two.

The octahedron which has eight triangular faces is often wrongly placed after the hexahedron which has six equal square faces or planes. However, when four tetrahedra are arranged together to form a bigger tetrahedron, one octahedral space is formed within (figure 1b). Thus logically the octahedral space can be said to have evolved from the association of four tetrahedra. With the octahedron inside four tetrahedra the addition of four tetrahedra on to the vacant sides of the octahedron results in the formation of a cube with eight vertices (figure 1c). A hexahedron is formed by covering the cube with six equal squares (figure 1d). The length of each square plane is about 4.5 cm if the base of each of the eight regular tetrahedra in 3 cm. Thus, eight tetrahedra with an octahedron within can be arranged within a hexahedron. The hexahedron has six faces made of equal squares (figure 1d).

Another solid which is readily formed from tetrahedron is the icosahedron which is a cluster of twenty tetrahedra and therefore has twenty triangular faces (figure 2a). This icosahedron can be transformed into a dodecahedron such that the latter encloses the former. The transformation of an icosahedron to a dodecahedron is illustrated in figure 2. This tranformation can be done by fixing a tetrahedron with equal triangular base on each of the twenty triangular faces on an icosahedron (figure 2b). Thirty tetrahedra spaces thus created among the first twenty tetrahedra are then filled with additional thirty tetrahedra (figure 2c). A dodecahedron is therefore formed as the next regular geometric solid to the hexahedron by sealing each of the twelve pentagonal faces with an equilateral pentagonal face (figure 2d).

ı) An icosahedron

ı) An icosahedron
ith 20 tetrahedra (toot-
ɜd icosahedron)

) An icosahedron with
ı tetrahedra (naked
ɔdecahedron)

) A dodecahedron (12
ntagonal faces)

Figure 2. Evolutionary stages of the dodecahedron from an icosahedron.

The octahedron and the icosahedron are also not completely regular or are less regular solids compared with the other three because of what is here described as presence of false planes in the octahedron and in the icosahedron (figure 3). A false plane consists of a major plane with a number of minor planes. A less regular solid symmetry arises when a major plane runs into other major and identical planes such that both or all major planes have one or more minor planes in common. Such major planes are here called siamese planes. As examples; two rectangular planes run into one another (siamese rectangular planes) with two triangular planes in common in an octahedron. Also siamese twin pentagonal planes with two triangular planes common to two pentagons, and siamese pentagonal triplet planes with four triangular planes common to three pentagons occur in an icosahedron.

The fact that both an octahedron can fit the interior of four or eight tetrahedra with equal triangular surface areas and that an icosahedron can fit the interior of a dodecahedron also suggests that nature abhor empty spaces or vacuums. The actual sequence in terms of evolutionary complexity of the five solids is illustrated in figure 3. The evolutionary sequence in terms of structural complexity of the three completely regular solids is shown in figure 4 asuming the octahedron occupies the interior of the hexahedron and that the icosahedron occupies the interior of a dodecahedron. The illustrations in figures 3 and 4 more or less represent the relative sizes of the different geometric solids as structural complexities increase from the tetrahedron to the dodecahedron.

Pedoe[4] gave an account of Kepler's efforts to separate these five geometric solids into two groups. Kepler[4] also grouped the tetrahedron, hexahedron and the dodecahedron together while the octahedron and icosahedron were also put in another group. This grouping is in agreement with the present work. Pedoe's[4] account did not cover Kepler's reasons for and method of this grouping. He, however, gave an account of Kepler's mathematico-religious interest in the geometry of solids in which Kepler's notion of an ordered creation of a model Universe was represented.

The number of edges arising from a vertex is three for the tetrahedron, the hexahedron and the dodecahedron but variable in the less regular solids, that is the octahedron and the icosahedron. The second equation of Euler[5], that is

$$nf = 2e = mv,$$

is true for the five solids described. A modification of

Solid	Shape	Number of faces	Number of true faces	Type of face
Tetrahedron		4	4	Triangles (true planes)
Octahedron		8	—	Triangles (false planes)
Hexahedron		6	6	Squares (true planes)
Icosahedron		20	—	Triangles (false planes)
Dodecahedron		12	12	Pentagons (true planes)

Figure 3. Type of faces in each of the five regular Solids.

this equation based on a constant of 3 edges arising from a vertex as below satisfies the shape of the more regular solids only. Thus all the relationships $nf = 2e = 3v$ are true for the tetrahedron, the hexahedron and the dodecahedron and no more.

The ability of simple geometric spaces to transform to complex ones as shown in this study suggests that similar evolutionary trends could have occurred in the formation of earth mineral elements and compounds, both of which display simple to complex geometric configurations. Transformations of mineral elements from simple to complex geometric structures could have occurred during nuclear fusion reactions within an active or living star. The formation of mineral compounds could have occurred within the cooling body of a dead star or that of a material that was gutted out as a by-product of nuclear fusion from a living star. Lowering of temperatures and pressures would allow the various chemical elements to interact to form chemical compounds. The compounds could also concentrate into fairly homogeneous materials of similar densities under the centrifugal and the centripetal forces to which such a cooling body would be subjected.

Solid	Shape	Number of faces	Number of sides to a face	Type of face
Tetrahedron		4	3	Triangles
Hexahedron		6	4	Squares
Dodecahedron		12	5	Pentagons

Figure 4. Number and type of faces in the three more regular Solids.

SUMMARY AND CONCLUSIONS

The empirical evidences provided in this study on the so-called five regular solids suggest that the regular tetrahedron, the regular hexahedron, and the regular dodecahedron with 4 triangular, 6 rectangular and 12 pentagonal faces, respectively are the completely regular and major solids. The other two, namely the octahedron and the icosahedron with 8 triangular and 20 triangular faces, respectively are less regular than the former three and are therefore minor solids. The external properties of the octahedron and the icosahedron respectively fit into the internal framework of the hexahedron and the dodecahedron.

This study suggests that existing mineral elements and chemical compounds within the earth's crust which also display simple to complex 3-D geometric patterns could have undergone a similar evolutionary pathway in the course of their formation. The widespread occurrence of these geometries in different disciplines, in the creative and mental abstractions of people regardless of time and space also suggests a need for a multi-disciplinary approach to their study. This will enhance the potential applications of the fundamental principles of nature that are inbuilt in the symmetry of these shapes.

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NEWS

CHELATION THERAPY: A MIRACLE CURE?

. . . "Chelation therapy, touted as a miracle cure by its proponents and denounced as a fraud by its critics, has been used to treat 400,000 people at 1,000 or so clinics across the nation, its proponents say . . . . The therapy involves the introduction of a chelating agent, ethylenediamine tetraacetate (EDTA), into a patient's bloodstream for about 20 to 30 four-hour-long sessions. The EDTA purportedly binds to, or chelates, the calcium in atherosclerotic plaques, breaking up the plaques and increasing the diameter of arteries to let blood flow through more easily. The EDTA-calcium complex is then excreted from the body. Such was the mechanism originally proposed in the mid-1950s to explain how chelation therapy works. But this mechanism, along with the therapy itself, has been discredited by most members of the medical profession, including the American Medical Assn., American Heart Assn., American Coll. of Physicians, and American Osteopathic Assn. An article in July 1984 *Harvard Medical School Health Letter* explains why: 'Even if chelation therapy did take calcium out of atherosclerotic plaques, it does not automatically follow that blood flow would improve as a result. The remaining material—cholesterol, excessive smooth muscle tissue and fibrous scar—would still remain to obstruct blood flow' . . . . Like laetrile, chelation therapy seems to be a modern version of patent medicines of old. No longer hawked on the streets, and more sophisticated than herbals and exotic mixtures, chelation therapy nonetheless appeals to those looking for a quick and easy fix to medical problems."

[(Dawn D. Bennett in *Science News* 127(9): 138-9, 2 Mar 85). Reproduced with permission from Press Digest, *Current Contents*®, No. 18, May 6, 1985, p. 14, (Published by the Institute for Scientific Information®, Philadelphia, PA, USA.)]