ON MHD UNSTEADY HELE-SHAW FLOW OF VISCO-ELASTIC FLUID—I

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1. INTRODUCTION

Lee and Fung, Buckmaster, Lamb and Thompson have discussed the steady Hele–Shaw flows of viscous incompressible fluids, assuming the pressure gradient to be constant. Swaminathan has investigated the unsteady Hele–Shaw flow of the viscous incompressible fluid taking the pressure gradient to be a function of time.

In this paper, we study the unsteady Hele–Shaw flow of visco-elastic fluid of Kuvshinskii type under the influence of uniform transverse magnetic field, assuming the pressure gradient to be proportional to \( \exp(-mt) \). We assume that the fluid is of small electrical conductivity with magnetic Reynolds number much less than unity so that the induced magnetic field can be neglected in comparison with the applied magnetic field (Sparrow and Cess). We have evaluated the velocity components in the Hele–Shaw cell and a number of results are deduced from them. We have investigated the effects of magnetic parameter \( M \) and the relaxation time \( \lambda \) on the velocity components \( u \) and \( v \). We have seen that the velocity component \( u \) decreases with the increase in \( M \) or \( \lambda \) whereas \( v \) increases with the increase in \( M \) or \( \lambda \).

2. FORMULATION AND SOLUTION OF THE PROBLEM

We consider the flow of a visco-elastic fluid confined between two parallel planes \( z = \pm d \) past a circular cylinder \( x^2 + y^2 = a^2 \), \(-d \leq z \leq d\) under the influence of uniform transverse magnetic field.

The equations governing the motion of the visco-elastic fluid in the Hele–Shaw cell under the influence of uniform transverse magnetic field (in the absence of any internal electric field) are

\[
\begin{align*}
0 &= -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial z} + \frac{\partial^2 u}{\partial x^2} - Mu \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{align*}
\]

(2.3)

(2.4)

where \( \lambda \) is the relaxation time, \( u \) and \( v \) are the velocity components along \( x \) and \( y \) axes, \( t \) the time, \( \rho \) the fluid density, \( p \) the pressure, \( \nu \) the kinematic coefficient of viscosity, \( \sigma \) the electrical conductivity, \( \mu_0 \) the magnetic permeability, \( H_0 \) the intensity of the magnetic field and

\[
M = \frac{\sigma \mu_0 a^2}{\rho},
\]

the magnetic parameter. Using (2.4), equations (2.1) and (2.2) yield

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = 0.
\]

(2.5)

Equation (2.3) implies that \( p \) is independent of \( z \).

Therefore \( p \) is a function of \( x, y \) and \( t \).

Suppose

\[
u = f(t, z) \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = f(t, z) \frac{\partial \phi}{\partial y}.
\]

(2.6)

where \( \phi \) is some function of \( x, y \). Substituting (2.6) in (2.4) we get

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]

(2.7)

From (2.1), (2.2) and (2.6) we obtain

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial x} = \rho \frac{\partial \phi}{\partial y} (vf_y - f_x - \lambda f_{yt} - Mf)\]

and

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial y} = \rho \frac{\partial \phi}{\partial x} (vf_x - f_y - \lambda f_{yt} - Mf).
\]

(2.8)

Integrating, we get

\[
\left(1 + \lambda \frac{\partial}{\partial t}\right) p = \rho \phi (vf_y - f_x - \lambda f_{yt} - Mf) + \rho \phi (t)
\]

(2.9)

where \( \phi (t) \) is an arbitrary function of time.

Let the pressure gradient be proportional to \( \exp(-mt) \).
Let
\[ v_{f a} - f_i - \lambda f_{\alpha n} - M f = - A \exp (- m t) \]  \hspace{1cm} (2.10)
where \( n \) is a positive integer and \( A \) is a given constant.

Suppose
\[ f(t, z) = \exp (- m t) F(z). \]  \hspace{1cm} (2.11)

Now the equation (2.10) becomes
\[ \frac{d^2 F}{dz^2} + \left( \frac{n - n^2 \lambda - M}{v} \right) F = - \frac{A}{v} \]  \hspace{1cm} (2.12)
The boundary conditions are
\[ F(z) = 0 \text{ on } z = \pm d. \]  \hspace{1cm} (2.13)

In view of (2.13), the solution of equation (2.12) is
\[ F(z) = - \frac{A}{(n - n^2 \lambda - M)} \left[ 1 - \frac{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z}{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z} \right]. \]  \hspace{1cm} (2.14)
The function \( \phi (x, y) \) can be evaluated by solving (2.7) subject to the condition \( u \cos \theta + v \sin \theta = 0 \) on \( r = a \)
or \[ \frac{\partial \phi}{\partial r} = 0 \text{ on } r = a \]  \hspace{1cm} (2.15)
where \( x = r \cos \theta, y = r \sin \theta \) and \( \frac{\partial \phi}{\partial x} \to 1, \frac{\partial \phi}{\partial y} \to 0 \) as \( |x|, |y| \to \infty \).

Hence
\[ \phi (x, y) = \left( r + \frac{a^2}{r} \right) \cos \theta. \]  \hspace{1cm} (2.16)

From equations (2.6), (2.11) and (2.14) we get velocity components
\[ u = - \frac{A \exp (- m t)}{(n - n^2 \lambda - M)} \left[ 1 - \frac{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z}{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)^2}{(x^2 + y^2)^2} \right]. \]  \hspace{1cm} (2.17)
\[ v = \frac{A \exp (- m t)}{(n - n^2 \lambda - M)} \left[ 1 - \frac{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z}{\cos \left( \frac{n - n^2 \lambda - M}{v} \right) z} \right] \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right]. \]  \hspace{1cm} (2.18)

Case (i): \( M = 0, \lambda \neq 0, n \neq 0 \)

The velocity components in the case of unsteady Hele-Shaw flow of visco-elastic fluid are
\[ u = - \frac{A \exp (- m t)}{n(1 - n \lambda)} \left[ 1 - \frac{\cos \left( \frac{n - n^2 \lambda}{v} \right) \frac{1}{2} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)^2}{(x^2 + y^2)^2} \right]. \]  \hspace{1cm} (2.19)
\[ v = \frac{A \exp (- m t)}{n(1 - n \lambda)} \left[ 1 - \frac{\cos \left( \frac{n - n^2 \lambda}{v} \right) \frac{1}{2} \right] \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right]. \]  \hspace{1cm} (2.20)
Case (ii): $M \neq 0$, $\lambda = 0$, $n \neq 0$

The velocity components in the case of unsteady Hele-Shaw flow of viscous, incompressible conducting fluid under the influence of uniform magnetic field are

$$u = - \frac{A \exp (-nt)}{(n-M)} \left[ 1 - \frac{\cos \left( \frac{n-M}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{n-M}{v} \right)^{\frac{1}{2}} d} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right].$$  

$$v = \frac{A \exp (-nt)}{(n-M)} \left[ 1 - \frac{\cos \left( \frac{n-M}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{n-M}{v} \right)^{\frac{1}{2}} d} \right] \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right].$$

Case (iii): $M \neq 0$, $\lambda \neq 0$, $n = 0$

The velocity components in the case of steady Hele-Shaw flow of visco-elastic fluid under the influence of uniform transverse magnetic field are

$$u = \frac{A^2}{M} \left[ 1 - \frac{\cos \left( \frac{M}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{M}{v} \right)^{\frac{1}{2}} d} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right].$$

$$v = - \frac{A^2}{M} \left[ 1 - \frac{\cos \left( \frac{M}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{M}{v} \right)^{\frac{1}{2}} d} \right] \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right].$$

Case (iv): $M = 0$, $\lambda = 0$, $n \neq 0$

The velocity components in the case of unsteady Hele-Shaw flow of viscous incompressible fluid are

$$u = - \frac{A \exp (-nt)}{n} \left[ 1 - \frac{\cos \left( \frac{n}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{n}{v} \right)^{\frac{1}{2}} d} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right].$$

$$v = \frac{A \exp (-nt)}{n} \left[ 1 - \frac{\cos \left( \frac{n}{v} \right)^{\frac{1}{2}} z}{\cos \left( \frac{n}{v} \right)^{\frac{1}{2}} d} \right] \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right].$$

Case (v): $M = 0$, $\lambda \neq 0$, $n = 0$

The velocity components in the case of steady Hele-Shaw flow of visco-elastic fluid are

$$u = - \frac{A (h^2 - z^2)}{2v} \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right].$$

$$v = \frac{A (h^2 - z^2)}{2v} \left[ \frac{2a^2 xy}{(x^2 + y^2)^2} \right].$$

It is interesting to note that these are the velocity components in the case of steady Hele-Shaw flow of viscous incompressible fluid.
Fig. 1. $u$ against $z$ for different $M$.

Fig. 2. $u$ against $z$ for different $\lambda$.

Fig. 3. $v$ against $z$ for different $M$.

Fig. 4. $v$ against $z$ for different $\lambda$. 
Case (vi): $M \neq 0, \lambda = 0, n = 0$

The velocity components in the case of steady Hele-Shaw flow of viscous, incompressible conducting fluid are

$$u = \frac{A}{M} \left[ 1 - \frac{\cos i (M \xi)^{\frac{1}{2}} z}{\cos i (M \xi)^{\frac{1}{2}} d} \right] \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right]$$

$$v = -\frac{A}{M} \left[ 1 - \frac{\cos i (M \xi)^{\frac{1}{2}} z}{\cos i (M \xi)^{\frac{1}{2}} d} \right] \frac{2a^2 xy}{(x^2 + y^2)^2}. \tag{2.29} \tag{2.30}$$

Case (vii): $M = 0, \lambda = 0, n = 0$

The velocity components in the case of steady Hele-Shaw flow of viscous incompressible fluid are

$$u = \frac{A}{2v} \frac{(z^2 - z^2)}{2v} \left[ 1 - \frac{a^2 (x^2 - y^2)}{(x^2 + y^2)^2} \right]$$

$$v = -\frac{A}{2v} \frac{(z^2 - z^2)}{2v} \frac{2a^2 xy}{(x^2 + y^2)^2}. \tag{2.31} \tag{2.32}$$

3. Conclusions

We have investigated the effects of magnetic parameter $M$ and the relaxation time $\lambda$ on the velocity components $u$ and $v$. In Fig. 1 the velocity component $u$ is plotted against $z$ for different values of $M$. It is clear from this figure that the velocity component $u$ decreases with the increase in magnetic parameter $M$. Figure 2 shows the effect of relaxation time $\lambda$ on velocity component $u$. We observe that $u$ decreases with the increase in $\lambda$. In Fig 3 $v$ is drawn against $z$ for different values of $M$ and for different values of $\lambda$ in Fig. 4. We notice that the velocity component $v$ increases as $M$ or $\lambda$ increases. We have derived velocity components $u$ and $v$ in different cases.


### PALYNOLOGICAL STUDIES ON SOME LIVERWORTS OF GARHWAL HIMALAYAS

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**Abstract**

The present communication pertains to the study of 20 species belonging to 13 genera of the order Anthocerotales and Marchantiales. Diversity in sporomorphs was noticed. The exine usually protrudes into various types of processes, i.e., verrucose, spinulose, spinose and gemmulate and represents various peculiar ornamentation such as foveolate, punctate and reticulate, indicating their distinctiveness. Perine and triradiate ridge may be present or absent.

**Introduction**

Although palynology has recently received due attention, there are only a few reports on the bryophytes. Our knowledge on the spores of bryophytes is mainly confined the general morphology and taxonomy of the group. Historical account on the spore study of bryophytes is given by Udare. However, some of the important contributions on different aspects were made by McClymont, Terasmae, Tallis, Erdman and others.

Perusal of earlier literature indicates great need for the study on spore morphology. The authors have made an attempt to study the spore morphology of some of the common liverworts collected from Garhwal Himalayas at the elevation range of 700-2000 m.