UNSTEADY, LAMINAR FLOW OF A VISCOUS INCOMPRESSIBLE FLUID THROUGH POROUS MEDIA IN A HEXAGONAL CHANNEL

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ABSTRACT

In this paper, unsteady laminar flow of a viscous incompressible fluid through porous media in a channel has been discussed. The cross-section of the channel is taken as a regular hexagon. The flow is initially at rest and it takes place when an arbitrary time varying pressure gradient is applied. Exact solution of the differential equation defining the flow has been obtained by using the technique of Finite Fourier and Laplace transforms. Various flows can be deduced from this for different values of $f(\tau)$. Effect of porosity on velocity profiles under constant pressure gradient has been represented by graphs.

1. INTRODUCTION

SEVERAL investigators[2-8] have discussed the flow of fluids through channels of various cross-sections, *i.e.*, rectangle, circle, ellipse, traingle, sector of a circle, annular sector, etc., but the flow through hexagonal channels has not yet been discussed. In the present paper, the flow through porous media in a regular hexagonal channel under the influence of arbitrary time varying pressure gradient has been discussed. Graphs have been drawn to report that (i) velocity at the points on the axis of the channel is maximum, (ii) velocity increases as porosity increases, (iii) steady state velocity in porous medium is less than in an ordinary medium (of full porosity), (iv) the time to reach the steady state in porous medium is less than that in an ordinary medium (of full porosity).

2. FORMULATION OF THE PROBLEM

Let us use the rectangular cartesian coordinates system (x, y, z) such that z-axis is along the axis of the channel and the cross-section of the channel is formed by the straight lines:

$$y = \pm \frac{(3)^{1/2} a}{2}, \; y + (3)^{1/2} x = \pm (3)^{1/2} a \text{ and}$$

$$y - (3)^{1/2} x = \pm (3)^{1/2} a. \tag{2.1}$$

The equations of motion (Ahmadi and Manvi[1]) in our case takes up the form

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) - \frac{\nu}{K} u_z \tag{2.2}$$

where u_z is the fluid velocity along the axis of the channel, p, ρ, μ and $\nu\,(=\mu/\rho)$ are the pressure, the density, the viscosity and the kinematic viscosity of the fluid respectively, K is the permeability of the medium and t is the time.

The initial and boundary conditions are:

$$\left.\begin{array}{l} \text{(i) } t \leqslant 0,\; u_z(x, y, t) = 0 \text{ everywhere in the channel} \\ \text{(ii) } t < 0,\; u_z(x, y, t) = 0 \text{ on the boundary of the channel given by (2.1)} \end{array}\right\} \tag{2.3}$$

Using the non-dimensional quantities,

$$x' = \frac{x}{a}, \; y' = \frac{y}{a}, \; z' = \frac{z}{a}, \; p' = \frac{pa^2}{\rho\nu^2},$$

$$u = \frac{u_z a}{\nu} \text{ and } \tau = \frac{t\nu}{a^2},$$

equation (2.2) transforms to

$$\frac{\partial u}{\partial \tau} = f(\tau) + \left(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2}\right) - \frac{1}{K_1} u \tag{2.4}$$

where $K_1 = K/a^2$ and $-\dfrac{\partial p'}{\partial z'} = f(\tau)$.

Now let $x_1 = y'$, $x_2 = y' - (3)^{1/2} x'$ and $x_3 = y' + (3)^{1/2} x'$. (2.5)

Under the transformations (2.5), equation (2.4) now become

$$\frac{\partial u}{\partial \tau} = f(\tau) + \left(\frac{\partial^2 u}{\partial x_1^2} + 4\frac{\partial^2 u}{\partial x_2^2} + 4\frac{\partial^2 u}{\partial x_3^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_3} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2} - 4\frac{\partial^2 u}{\partial x_1 \partial x_3}\right) - \frac{1}{k_1} u \tag{2.6}$$

and the conditions (2.3) become

(i) $\tau \leqslant 0$, $u(x_1, x_2, x_3, \tau) = 0$ everywhere in the channel (2.7)

$$\left.\begin{array}{l} \text{(ii) } \tau > 0,\; u(x_1, x_2, x_3, \tau) = 0 \text{ at} \\ x_1 = \pm (3)^{1/2}/2\,; \\ x_2 = \pm (3)^{1/2} \text{ and} \\ x_3 = \pm (3)^{1/2}. \end{array}\right\} \tag{2.8}$$

Due to the symmetry of motion about $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, we shall consider the motion in the region $x_1 \geqslant 0$, $x_2 \geqslant 0$ and $x_3 \geqslant 0$ and accordingly the boundary conditions (2.8) become

$$u(x_1, x_2, x_3, \tau) = 0 \text{ at } x_1 = 0, (3)^{1/2}/2; \; x_2 = 0, (3)^{1/2} \text{ and } x_3 = 0, (3)^{1/2} \tag{2.9}$$

and

$$\frac{\partial u}{\partial x_1} = 0 \text{ at } x_1 = 0, \; \frac{\partial u}{\partial x_2} = 0 \text{ at } x_2 = 0 \text{ and}$$

$$\frac{\partial u}{\partial x_3} = 0 \text{ at } x_3 = 0,$$

3. Solution

To solve equation (2.6) under the conditions (2.7) and (2.9), we shall make use of the following transforms:

(i) Finite Fourier Cosine Transform with respect to x_1, defined as

$$\bar{u}_1(p, x_2, x_3, \tau) = \int_0^{(3)^{1/2}/2} u(x_1, x_2, x_3, \tau) \cos P_p x_1 dx_1 \tag{3.1}$$

where $P_p = \dfrac{(2p+1)\pi}{(3)^{1/2}}$

under the conditions:

$$\left.\begin{aligned} &u_1(x, x_2, x_3, \tau) = 0 \text{ at } x_1 \\ &\quad = 0 \text{ and } (3)^{1/2}/2 \\ &\text{and } \frac{\partial u}{\partial x_1} = 0 \text{ at } x_1 = 0 \end{aligned}\right\} \text{ for } \tau > 0.$$

(ii) Finite Fourier Cosine Transform with respect to x_2, defined as

$$\bar{u}_2(p, q, x_3, \tau) = \int_0^{(3)^{1/2}} \bar{u}_1(p, x_2, x_3, \tau) \cos Q_q x_2 dx_2 \tag{3.2}$$

where $Q_q = \dfrac{(2q+1)\pi}{2(3)^{1/2}}$

under the conditions:

$$\left.\begin{aligned} &\bar{u}_1(p, x_2, x_3, \tau) = 0 \text{ at } x_2 = 0 \text{ and } (3)^{1/2} \\ &\text{and} \quad \frac{\partial u}{\partial x_2} = 0 \text{ at } x_2 = 0 \end{aligned}\right\} \text{ for } \tau > 0$$

(iii) Finite Fourier Cosine Transform with respect to x_3, defined as

$$\bar{u}_3(p, q, r, \tau) = \int_0^{(3)^{1/2}} \bar{u}_2(p, q, x_3, \tau) \cos R_r x_3 dx_3 \tag{3.3}$$

where

$$R_r = \frac{(2r+1)\pi}{2(3)^{1/2}}$$

under the conditions:

$$\left.\begin{aligned} &\bar{u}_2(p, q, x_3, \tau) = 0 \text{ at } x_3 = 0 \text{ and } (3)^{1/2} \\ &\text{and} \quad \frac{\partial u}{\partial x_3} = 0 \text{ at } x_3 = 0 \end{aligned}\right\} \text{ for } \tau > 0.$$

(iv) Laplace Transform with respect to τ, defined as

$$\bar{u}(s) = \int_0^{\infty} \bar{u}_3(p, q, r, \tau) e^{-s\tau} d\tau \tag{3.4}$$

under the condition:

$$\bar{u}_3(p, q, r, \tau) = 0 \text{ for } \tau = 0.$$

It is to be noted that on account of u being an even function of x_1, x_2 and x_3, the Finite Fourier Sine Transforms will be zero.

Now taking the Finite Fourier Cosine Transforms of equation (2.6) with respect to x_1, x_2 and x_3 in succession, we obtain

$$\frac{\partial \bar{u}_3}{\partial \tau} = \frac{(-1)^{p+q+r}}{P_p Q_q R_r} f(\tau) - (P_p^2 + 4Q_q^2 + 4R_r^2)\bar{u}_3 - \frac{1}{K_1}\bar{u}_3. \tag{3.5}$$

Taking the Laplace transform of equation (3.5), we get

$$\bar{u}(s) = \frac{(-1)^{p+q+r}}{P_p Q_q R_r} \frac{\bar{f}(s)}{(s+\alpha)} \tag{3.6}$$

where $\bar{f}(s)$ is the Laplace transform of $f(\tau)$ and

$$\alpha = P_p^2 + 4Q_q^2 + 4R_r^2 + \frac{1}{K_1} \tag{3.7}$$

Inverting the Laplace transform by Convolution theorem,

$$\bar{u}_3 = \frac{(-1)^{p+q+r}}{P_p Q_q R_r} \int_0^{\tau} e^{-\alpha\lambda} f(\tau - \lambda) d\lambda \tag{3.8}$$

Inverting the Finite Fourier Cosine Transforms with respect to x_3, x_2 and x_1 in succession, we finally obtain

$$u = \frac{(2)^4}{(3)^{3/2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r}}{P_p Q_q R_r} \times \left\{ \int_0^{\tau} e^{-\alpha\lambda} f(\tau - \lambda) d\lambda \right\} \times \cos P_p x_1 \cos Q_q x_2 \cos R_r x_3 \tag{3.9}$$

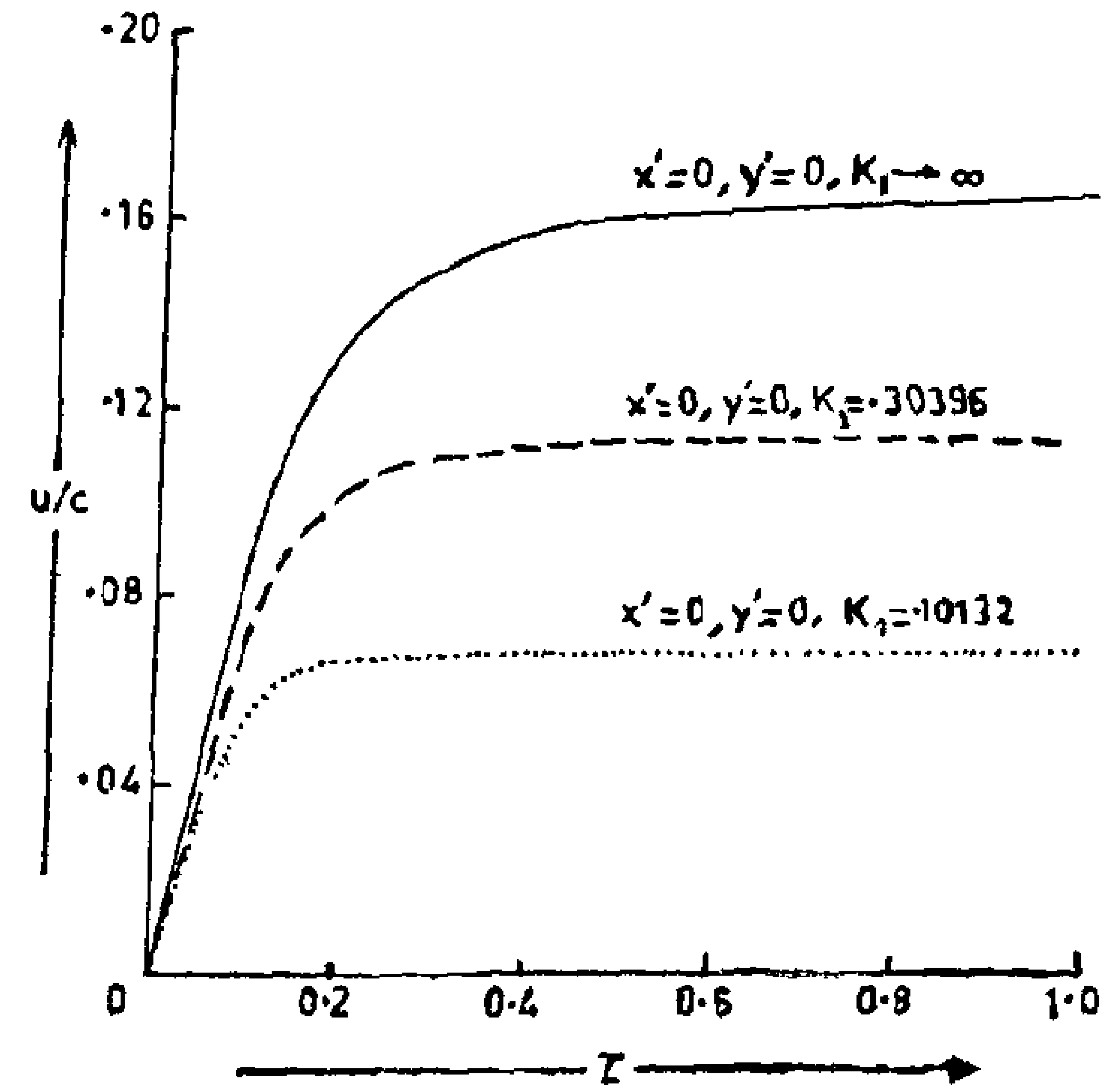

Fig. 1. Velocity plotted against τ for different values of K_1, at $x' = 0$, $y' = 0$ and $f(\tau) = C$.

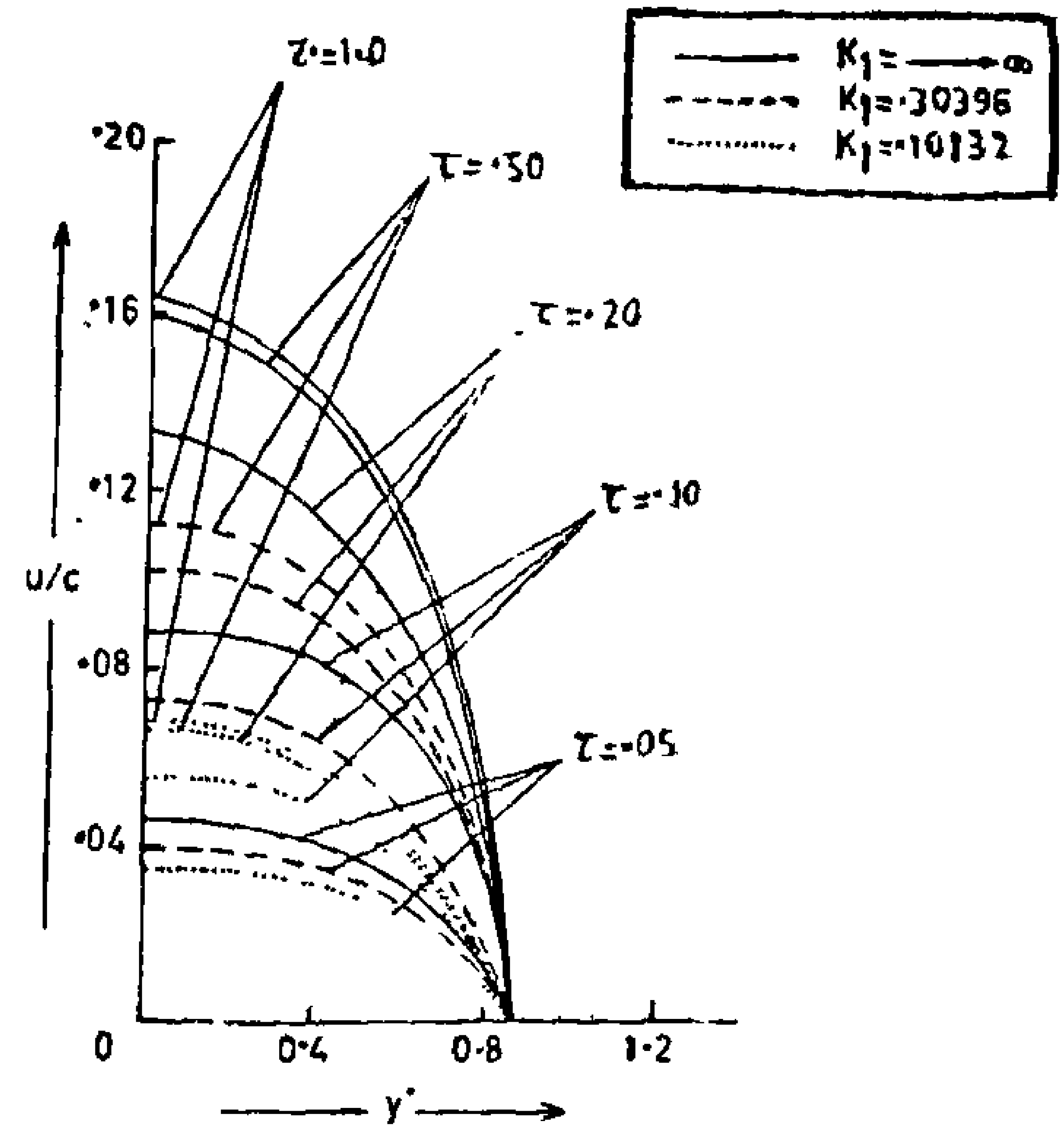

Fig. 2. Velocity plotted against y' for different values of τ and K_1 at $x' = 0$, and $f(\tau) = C$.

4. Particular Case : Constant Pressure Gradient

Here $f(\tau) = C$, where C is an absolute constant. Substituting this value of $f(\tau)$ in equation (3.9), we have

$$\frac{u}{C} = \frac{(2)^4}{(3)^{3/2}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{p+q+r}}{P_p Q_q R_r} \frac{(1 - e^{-a\tau})}{a} \times \cos P_p x_1 \cos Q_q x_2 \cos R_r x_3 \tag{4.1}$$

when $K_1 \to \infty$ the resistancy of the porous medium becomes negligible *i.e.*, the medium becomes an ordinary medium (full porosity). Thus if we put $K_1 \to \infty$ in equation (3.9) and (4.1) we immediately get the velocity distribution for the flow in an ordinary medium under arbitrary time varying pressure gradient and constant pressure gradient, respectively.

Discussion

Various flows for different values of pressure gradient can be deduced from equation (3.9) Equation (4.1) gives the velocity distribution under constant pressure gradient. Figs. 1 and 2 report that (i) velocity at the points on the axis of the channel is maximum, (ii) velocity increases as porosity increases, (iii) steady state velocity in porous medium is less than that in an ordinary medium (of full porosity), (iv) the time to reach the steady state in porous medium is less than that in an ordinary medium (of full porosity).

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