MATHEMATICAL SOLUTION FOR THE FLOW OF TWO IMMISCIBLE FLUIDS IN A DIPPING CRACKED POROUS MEDIUM WITH MEAN PRESSURE

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ABSTRACT

This paper presents an analytical discussion of the flow of two immiscible liquids in dipping cracked porous medium. The problem has much importance in petroleum technology. An attempt has been made to include capillary pressure effect in the analysis and it is found that the mathematical formulation yields a nonlinear differential equation. This equation of motion has been solved by the perturbation technique to give an analytical expression for the displacing phase saturation distribution under the simplifying assumption that the individual pressure of the two phases may be replaced by their common mean pressure which is regarded as a constant.

INTRODUCTION

THE oil water movement in a cracked porous medium is a problem of petroleum technology and hydrogeology. Many authors have discussed this problem from different viewpoints [Bokserman, Zheltov and Kocheshkov (BZK)[1], Bear and Braester[2], Mattax and Kyte[3] and Verma[4-6]]. Most of the authors except Verma have completely neglected the capillary pressure effect.

In this paper the underlying assumption is made that the individual pressure of the two flowing phases may be replaced by their common mean pressure and expression for the phase saturation is determined. The solution of the nonlinear differential equation has been obtained by perturbation method under the simplifying assumption.

We consider here that water is uniformly injected with a constant velocity into an oil saturation, dipping bed of homogeneous medium traversed by a branched system of cracks varying in orientation. It is assumed that the entire oil on the initial boundary ($x = 0$) is displaced through a small distance due to the impact of injected water.

The main interest of the present investigation is to obtain an analytical expression for phase saturation when the porosity and permeability of the medium may be regarded as constant. The mathematical solution of the nonlinear differential equation of the flow of two immiscible fluids may be obtained under certain special conditions.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The seepage velocity of oil (V_0) and water (V_w) may be written by Darcy's law as

$$V_0 = -\frac{K_0}{\delta_0} K \left(\frac{\partial p_0}{\partial x} + \rho_0 g \sin \alpha\right), \tag{2.1}$$

$$V_w = -\frac{K_w}{\delta_w} K \left(\frac{\partial p_w}{\partial x} + \rho_w g \sin \alpha\right). \tag{2.2}$$

The equation of continuity for the flow of water and oil in the cracked porous medium may be written as

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} + \phi\,[T - \tau(u)] = 0, \tag{2.3}$$

$$P \frac{\partial S_0}{\partial t} + \frac{\partial V_0}{\partial x} - \phi\,[T - \tau(u)] = 0 \tag{2.4}$$

where $\phi\,[T - \tau(u)]$ is the impregnation function of the crack system. It is the amount of displacing liquid entering the blocks in an elementary volume of the seam and is analytically defined, following Mattax and Kyte[3] (as in BZK[1]), and Verma[5] by the expression

$$\phi\,[T - \tau(u)] = D\,(T - Rx^2)^{-1/2},\ \tau \leqslant t$$

$$u = x/L_M,\ T = \theta t,\ R = a/L_M^2$$

$$t = \frac{\delta \cos \theta\, S^2 (K/m_B)^{1/2}}{\delta_0} \tag{2.5}$$

$$a = \left(\frac{\pi A}{4q} \cdot \frac{S^2\, \delta \cos \theta\, m_B g_A (K/m_B)^{1/2}}{\delta_0}\right)^2$$

From the definition of phase saturation (Scheidegger[7]) we have

$$S_w + S_0 = 1. \tag{2.6}$$

The capillary pressure (p_c) in a double phase fluid flow is defined to be the pressure discontinuity between the flowing phases across their common interface, and may be written as

$$p_c = p_0 - p_w \tag{2.7}$$

combining equations (2.1) to (2.4) and using equations (2.6) and (2.7), we get

$$\frac{\partial}{\partial x}\left[\left(\frac{K_w}{\delta_w} + \frac{K_0}{\delta_0}\right) K \frac{\partial p_0}{\partial x} - \frac{K_w}{\delta_w} K \frac{\partial p_c}{\partial x} + \left(\frac{K_w}{\delta_w} K \rho_w + \frac{K_0}{\delta_0} K \rho_0\right) g \sin \alpha\right] = 0 \tag{2.8}$$

Integrating equation (2.8) with respect to x and by simplifying it we get

$$\frac{\partial P_0}{\partial x} = -\frac{V}{(K_w \delta_w - K_0 \delta_0) K} - \frac{1}{(1 + K_0\delta_w K_w\delta_0)} \times \frac{\partial P_c}{\partial x} - \frac{\rho_w - \rho_0(K_0\delta_w K_w\delta_0)}{(1 + K_0\delta_w K_w\delta_0)} g \sin\alpha \quad (2.9)$$

where V is a constant of integration, which can be evaluated later on.

The value of the pressure of oil (p_0) can be written as (Oroveanu[8])

$$p_0 = (p_0 + p_w)/2 + (p_0 - p_w)/2 = \bar{p} + p_c/2 \quad (2.10)$$

where $\bar{p}$ is the mean pressure which is constant.

Substituting the value of $\partial p_0/\partial x$ from equation (2.10) into eq. (2.9), we get

$$V = \frac{1}{2}\left(\frac{K_w}{\delta_w} K - \frac{K_0}{\delta_0} K\right)\frac{\partial p_c}{\partial x} - \left(\frac{K_w}{\delta_w} K\rho_w + \frac{K_0}{\delta_0} K\rho_0\right) g \sin\alpha \quad (2.11)$$

Combining equations (2.3), (2.9) and substituting the value of V and $\phi\,[T - \tau(u)]$ from equations (2.11) and (2.5), we obtain

$$\epsilon P \frac{\partial S_w}{\partial T} + \frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w} K\left\{\left(\frac{1}{2}\right)\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} + \rho_w g \sin\alpha\right\}\right] = -\frac{D}{\sqrt{(T - Rx^2)}}. \quad (2.12)$$

This is the nonlinear differential equation of motion for saturation of two immiscible fluids in dipping cracked porous medium when the capillary pressure is present.

Solution of the Problem

It is very difficult to obtain the exact solution of the equation (2.12) due to the nonlinear terms occurring therein. The underlying assumption is that ϵP is small for cracked porous medium of (BZK) because, in this case, $\epsilon \sim 10^{-4}$ to 10^{-6} and $P \sim 10^{-2}$ to 10^{-3}. Therefore, we obtain here an approximate analytical solution of eq. (2.12) by using a perturbation technique[10] (as in Verma[11]), and regarding ϵP as perturbation parameter (as in Verma[5]). Thus, neglecting $(\epsilon P)\,\partial S_w/\partial T$ at the first step, we get

$$\frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w} K\left\{\left(\frac{1}{2}\right)\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} + \rho_w g \sin\alpha\right\}\right] = -\frac{D}{\sqrt{(T - Rx^2)}} \quad (2.13)$$

Performing the integration with respect to x, we get

$$\frac{K_w}{\delta_w} K\left[\left(\frac{1}{2}\right)\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} + \rho_w g \sin\alpha\right] = -\frac{D}{\sqrt{R}}\sin^{-1}(x\sqrt{R/T}) + F(T) \quad (2.14)$$

where F(T) is the constant of integration which is determined by using the condition

$$\partial S_w/\partial x = 0, \quad \text{at } x = L \text{ for all time} \quad (2.15)$$

Substituting equation (2.15) into (2.14), we get

$$\frac{K_w}{\delta_w} K\left[\frac{1}{2}\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} + \rho_w g \sin\alpha\right] = -\frac{D}{\sqrt{R}}\sin^{-1}(x\sqrt{R/t}) + \frac{K_w}{\delta_w} K\rho_w \times g \sin\alpha + \frac{D}{\sqrt{R}}\sin^{-1}(L\sqrt{R/T}). \quad (2.16)$$

At this stage, we assume for definiteness some specific relationships which are quoted in standard literature.

$$K_w = S_w^3, \quad K_0 = 1 - \alpha S_w, \quad \alpha = 1\cdot 11$$
$$p_c = \beta(S_w^{-1} - C)\ (\beta \text{ and C are constants}). \quad (2.17)$$

The first one is due to Scheidegger[7] and the second is given by Mishra[9].

Substituting these values in equation (2.16), we get

$$S_w\frac{\partial S_w}{\partial x} = \frac{2D\delta_w}{\sqrt{R}\,K\beta}\left[\sin^{-1}(x\sqrt{R/T}) - \sin^{-1}(L\sqrt{R/T})\right]. \quad (2.18)$$

Integrating again the equation (2.18) with respect to x, we get

$$\frac{\beta K\sqrt{R}}{4\delta_w D} S_w^2 = x\sin^{-1}(x\sqrt{R/T}) - x\sin^{-1}(L\sqrt{R/T}) + \sqrt{(T/R - x^2)} + E(T). \quad (2.19)$$

where E(T) is the constant of integration which may be evaluated from the following condition:

$$S_w = 1/\alpha \quad \text{at} \quad x = 0. \quad (2.20)$$

Then combining equations (2.19) and (2.20), we get

$$S_w^2 = 1/\alpha^2 + \frac{\sqrt{R}\,\beta K}{4\delta_w D}\left[x\sin^{-1}(x\sqrt{R/T}) - x\sin^{-1}(L\sqrt{R/T}) + \sqrt{(T/R - x^2)} - \sqrt{T/R}\right]. \quad (2.21)$$

The next step in the perturbation method is to find the value of $\partial S_w/\partial T$ from equation (2.21) and substitute it in the first term of equation (2.12).

$$\frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w} K\left(\frac{1}{2}\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} + \rho_w g \sin\alpha\right)\right] = -\frac{D}{\sqrt{(T - Rx^2)}} + \left(\frac{1}{2}\right)\epsilon P\sqrt{\left(\frac{\beta K R^{3/2}}{4\delta_w D}\right)} \times \frac{x^2}{T^{3/2}}. \quad (2.22)$$

Integrating equation (2.22) with respect to x under the condition (2.15), we get

$$\frac{1}{2}\frac{K_w}{\delta_w} K\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x} = \frac{D}{\sqrt{R}}\left[\sin^{-1}(L\sqrt{R/T}) - \sin^{-1}(x\sqrt{R/T})\right] + \left(\frac{1}{6}\right)\epsilon P \times \sqrt{\left(\frac{\beta K R^{3/2}}{4\delta_w D T^3}\right)}(x^3 - L^3) \quad (2.23)$$

substituting the values of K_w, K_0 and p_c from eq. (2.17) in eq. (2.23) and integrating with respect to x under the condition (2.20), we get

$$S_w^2 = (1/2a^2) + \frac{4\delta_w}{K\beta}\left[\frac{D}{\sqrt{R}}(x\sin^{-1})(L\sqrt{R/T}) - x\sin^{-1}(x\sqrt{R/T}) + \sqrt{(T/R - x^2)} - \sqrt{T/R} + (1/6)\,\epsilon P\sqrt{\left(\frac{\beta K R^{3/2}}{4\delta_w D T^3}\right)} \times (x^4/4 - L^3 x)\right] \tag{2.24}$$

This is the required solution for the flow of two immiscible fluids in dipping porous medium.

Conclusion

In this paper we have obtained the approximate analytical solution of the flow of two immiscible liquids in a dipping cracked porous medium by perturbation method, under certain conditions. The underlying assumption is that the individual pressure of two flowing phases may be replaced by their common mean pressure, some specific results (which are quoted in standard literature) for relative permeability, capillary pressure, impregnation function have been considered in the present discussion. The problem has gained much current importance in petroleum technology and hydrogeology. The saturation of the water is determined. We have not included any numerical or graphical illustration due to our particular interest in an analytical solution.

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Nomenclature

A = Constant;
C = Capillary pressure constant;
K = Permeability of the medium;
K_w = Relative permeability of water ;
K_0 = Relative permeability of oil;
L = Length of the cylindrical porous matrix;
L_u = Mean block size;
m_u = Porosity of the block;
P = Porosity of the crack system;
p_w = Pressure of the water;
p_0 = Pressure of the oil;
q = Average rate of flow in across the face;
S = Mean specific surface area of the blocks;
S_w = Saturation of the water;
S_0 = Saturation of the oil;
t = Time;
T = Weighted time;
u = Mean coordinate;
V_w = Seepage velocity of water;
V_0 = Seepage velocity of oil;
x = Linear coordinate;
β = Capillary pressure coefficient;
ϕ = Impregnation function
δ = Surface tension;
δ_w = Viscosity of the water;
δ_0 = Viscosity of the oil;
g_k = Saturation of blocks with water at the moment t_k;
ϵ = A complex constant of cracked characteristics and the viscosity of oil;
θ = Angle of wetting.
$\tau < t$

Those symbols which are defined in the text are not included in this table.

1. Bokserman, A. A., Zheltov, P. Yu. and Kocheshkov, A. A., *Soviet Phys. Doklady*, 1964, **9**, 285.
2. Bear, J. and Braester, C., *Int. Symp. Fund. Trans. Phenom.*, Porous Media III Haifa, 1969, pp 21.
3. Mattax, C. C. and Kyte, I. R., *Petroleum Engg. J.*, 1962, p. 2.
4. Verma, A. P., *Rev. Roum. Sci. Techn. Mec. Appli.*, 1968, p. 277.
5. —, *Can. J. Phys.*, 1969, **47**, 2519.
6. —, *Proc. Nat. Inst. of Science of India*, 1969, **3**, 35.
7. Scheidegger, A. E., *The Physics of Flow through Porous Media*, University of Toronto Press, Toronto, 1960, **216**, 224.
8. Oroveanu, T., (Schuragerea Fluide'or Multifazice Prin Medii Poroase), Ed. *Acad. Bucharest*, 1966, **324**, 386.
9. Mishra, S. K. and Verma, A. P., *Indian J. of Pure and Applied Mathematics*, accepted for publication.
10. Bellmar, R., *Perturbation Techniques in Mathematics, Physics and Engineering*, Holt, Rinehart and Winston, Inc., New York, 1966, p. 2519.
11. Verma, A. P., *Can. J. Phys.*, 1969, **47**, 319.