IMAGE RECONSTRUCTION USING FRACTIONAL FOURIER TRANSFORMS

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ABSTRACT

A method is proposed for a consistent analysis of many images' of an object in any plane of observation between the diffraction plane and gaussian image plane of an image forming system, after Wiener-Condon-Patterson's approaches of expressing a Fourier transform as a subgroup of a continuous transformation group.

THE knowledge of the mechanism of image formation and the methods of their reconstruction has been enriched during the last decade. However, some problems are still to be solved, especially in the field of electron microscopy. Image reconstruction methods use many manipulations in the reciprocal or direct space[1,2]. The image itself is then synthesised in a digital or analog computer[3]. Due to a number of optical and phase problems the reconstructed image may not represent those features recorded in the original specimen or micrograph[4]. In electron microscopic objective aperture limited, imaging of objects of normal thickness and even with thin objects with paracrystalline or short range order, it is known that defocused and gaussian focussed images are complicated by the simultaneous existence of positive and negative phase contrasts as well as diffraction contrast[5-7]. In practical electron microscopy, there is a tendency to underfocus the image to within certain limits which will allow contrast to be enhanced by phase contrast. Such defocusing results in artefacts and contrast reversals in the final image[8]. General theory of image formation in optical instruments also raises some doubts whether the electron micrographs showing fine details and fringes representing lattice planes in crystals represent true images[9]. 'Image' off the focal plane equals the Fresnel image of a lensless aperture of suitable size[10]. Such patterns are believed to arise from Fresnel diffraction. For the defocus $\Delta f > 100$ nm Fresnel diffraction is a dominant feature of the image. However, Fresnel diffraction is usually neglected in the theory of image formation. One wonders whether this readily visible but normally discarded data could not be used for scientific purposes. If it is so, their rationalization could considerably extend the use of image reconstruction.

According to Abbe theory of image formation[11], if a cross-grating is located to the left of the first focal plane of a convergent lens and illuminated by a parallel beam of light then the diffraction pattern or the Fourier transform of the grating will be imaged in the second focal plane while the grating itself will be imaged in the gaussian image plane to the right of the second focal plane and is considered to be the Fourier transform of the diffraction pattern. It is apparent that the diffraction pattern, gaussian image and the 'image' intensity distribution in any plane between the focal plane and the gaussian image plane essentially contain the same information, the difference being the manner in which this information is displayed. The problem of expressing the 'image' intensity in any plane between the focal plane and the gaussian image plane as a transform of the intensity in any other such plane requires a solution in terms of Wiener[12], Condon[13] and Patterson[14] continuous transforms. The operation of Fourier transform generates a cyclic group of order 4 which is isomorphic with the group of rotations of a plane about a fixed point through integral multiples of a right angle[13]. Wiener-Condon-Patterson formulation provides a solution to the problem of embedding the Fourier transform group of order 4 in a continuous group, and a general expression for the kernels of integral transforms which leads to sets of functional spaces lying between any given function space and its Fourier transform space. One may thus visualize fraction Fourier transforms and fractional spaces, ordinary Fourier transform and reciprocal space being particular cases.

Many investigators[15-18] have described procedures in which a series of electron micrographs at different focii are taken and evaluated or image is reconstructed by a combined analysis of gaussian image and diffraction pattern. Here a theoretical discussion is given of some of the possibilities for image reconstruction and evaluation from out of focus images. The principal feature of the method, which is based on fractional Fourier analysis of images, is that the information lost in one image can be replaced by information on other images with different defocusing parameters. This also provides the possibility of a combined phase and diffraction contrast analysis of images, thereby leading to better resolved images or removing the ambiguities in the reconstructed images. After Patterson[14], the generalized transform may be expressed as:

$$G(\vec{u}, m_1) = \int K(\vec{u}, \vec{x}, dm) \,.\, F(\vec{x}, m_2) \,.\, d\vec{x} \qquad (1)$$

where

$$\begin{aligned} K(\vec{u}, \vec{x}, dm) &= (Q^{1/2}\, a/(\pi(1-t^2))^{1/2}) \,.\, \exp a^2 \big[(2t/(1-t^2)) \\ &\quad \times (\vec{u} \,.\, \vec{x}) - ((1+t^2)/2(1-t^2)) \\ &\quad \times (\vec{x} \,.\, \vec{x} + \vec{u} \,.\, \vec{u})\big] \end{aligned} \qquad (2)$$

and

$$\begin{aligned} a &= \sqrt{2\pi}(1-\beta^4)^{1/4}; \quad Q = \sqrt{1+\beta^4} - \beta^2; \\ dm &= m_1 - m_2; \quad t = Q \cdot i^{dm} = Q \cdot \exp(i\phi); \\ \phi &= dm\,\pi/2; \quad \beta \to 0. \end{aligned} \tag{3}$$

m_1 and m_2 are parameters corresponding to various observation planes. The kernel K, in the limit $\beta \to 0$, approaches a delta function for $dm = 0$ and 2 and Fourier transform pairs for $dm = 1$ and 3 and generalized inverse pairs when $m_1 + m_2 = 4$. Fractional values between 1 and 2 for dm corresponds to space transforms between diffraction and image planes.

A simple scheme for image correction and reconstruction will require at least two defocused images, say G's, corresponding to the observation planes m_1 and m_2, related by

$$G(\vec{x}, m_1) = \int G(\vec{u}, m_2) \cdot K(\vec{u}, \vec{x}, dm) \cdot d\vec{u} \tag{4}$$

which can be solved for dm numerically. Final reconstructed image should be consistent with the following relations, representing 'images' in other observation planes,

$$\begin{aligned} F(\vec{x}, 0) &= \int G(\vec{u}, m_i) \cdot K(\vec{u}, \vec{x}, m_i) \cdot d\vec{u} \\ &= \int G(\vec{v}, m_j) \cdot K(\vec{v}, \vec{x}, m_j) \cdot d\vec{v} \\ &= \int G(\vec{v}, m_j) \cdot K(\vec{v}, \vec{u}, dm) \cdot d\vec{v} \end{aligned} \tag{5}$$

An estimate for m's can be obtained from the defocus relation

$$df = \tan\phi \tag{6}$$

Kernel of transformation for certain ranges and values of m or ϕ can be easily shown to represent the formation of Fresnel and Fourier images[19,20]. Under such conditions Kernel K is formally equivalent to the kernel for Fresnel transform of the given function, a limiting case of which is the Fourier transform. Fresnel transform is also an unambiguous representation of the function and inversion reconstructs the original function, however, it rarely resembles the original function[21,22]. Although there are some problems with the behaviour of K for very small values of dm[14], the existence of Fresnel transform is assured if the Fourier transform of the function exists. In the range of our interest ($1 \leqslant m \leqslant 2$) and somewhat larger values of dm, initial computations to explore the feasibility of the method, with one dimensional functions, have been carried out, the details of which will be published elsewhere. Here it should be mentioned that the present approach provides a unified view of defocused images and images with wide focal range can be interpreted in a systematic way. Similar to the work of Gerchberg and Saxton[23,24], who have studied the relationships implied by Fourier theorem between the wave amplitudes and the wave phases at the back-focal plane of the objective lens and image plane, by the present method the complete wave function in the diffraction plane may be determined from a consistent analysis of the observed intensity distribution of the two or more defocused images. Such fractional images are in general complex[25] but in certain cases are useful in phase recovery[26]. Many complex patterns observed with convergent electron beams[27] may contain such images. The explanation suggested above for the appearance of fractional images and their analysis for reconstructing images is a tentative one, and requires further detailed study, the basic phenomenon is, however, readily understood. Attempts are being made to apply the method to the reconstruction of images from electron microscope.

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