Improved criteria on robust analysis for linear system using convex combination and geometric sequence methods

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This article addresses the robust analysis on a delayed system with uncertainties. A geometric sequence division (GSD) method is applied for delay partition. Then, a GSD-dependent Lyapunov–Krasovskii functional (LKF) is newly proposed, in which the integral interval relevant with the state variables forms in geometric progression. In addition, by applying the convex combination method, parameter uncertainties and the delay derivative \( \dot{d}(t) \) can thus be flexibly overcome. As a result, unnecessary enlargement for estimating the LKF derivative is eliminated. Numerical example shows that this proposed work achieves expected results.

Keywords: Convex combination, delay partition, geometric sequence, parameter uncertainties.

In the real world, delay is inevitably experienced in dynamic systems, such as chemical reaction processes, biological systems, mechanical systems, etc. The existence of delay can often yield poor performance or even instability. Therefore, how to overcome the negative influence of delays has attracted vast attention in recent academic research. Meanwhile, stability is considered as the priority in many applications, a lot of research has been done on stability analysis for various types of delayed processes in recent decades[2-5]. Generally, stability conditions of delayed systems are categorized as delay-independent and delay-dependent ones. Less conservative results can be obtained using delay-dependent conditions in case of a relatively small delay or interval time-varying delay.

For stability analysis, the essential issue for a system with delay is to obtain maximum delay upper bound that guarantees the studied system to be asymptotically stable. For obtaining a higher delay upper bound, various forms of Lyapunov–Krasovskii functional (LKF) are constructed, such as discretized LKF, augmented LKF and delay-partitioning LKF[6-10]. In fact, expected stability results will be achieved if well-developed inequalities are employed for estimating LKF derivative. Therefore, estimation of LKF derivative alternatively for reducing conservatism is another considerable option. Three techniques are commonly applied: Jensen-inequality[11-13], free weight matrix methods[14,15], convex optimization method including their combinations[16-21]. The well known Jensen-inequality and its modification are proposed in refs 18 and 19. However, as a result of handling LKF derivative, some terms are neglected. So, it leads to conservative conditions because of the estimation. How to avoid unnecessary conservatism in the estimation of LKF derivative is still a challenge.

In addition, uncertain dynamic behaviours commonly exist in practical implementation due to modelling errors, immeasurable issues and perturbations, which could degrade the performance of system or even cause system instability[22,23]. The analysis of uncertainties is another hot topic for studying dynamic systems[24-27]. However, reducing conservatism in such systems is normally accompanied with extra computational complexity. How to compromise conservative reduction and computation burden is also full of challenges.

In this article, stability analysis on an uncertain system is studied. Delay-dependent criteria are derived to ensure that the linear system is globally asymptotically stable under the maximum upper bound. Comparing with existing results, the main outcome would be as follows:

1) The recently developed geometric sequence division (GSD) method is first applied on time-varying delay in a linear system. Using this method, a new modified LKF is constructed, which contains new GSD-dependent integral forms with unfixed intervals. This approach can dramatically reduce the number of decision variables. This new development can provide expected stability conditions with high efficiency and less computational complexity.

2) Convex combination method is introduced to represent the parameter uncertainties and solve the delay derivative. This estimation approach can handle the LKF derivative without using extra inequalities or constraint conditions. Thus, unnecessary enlargement can be eliminated.
Notations: \( \mathbb{R}^n \) is the Euclidean space with \( n \)-dimensional. \( P > (\geq) 0 \) indicates positive (semi-positive) definite matrix \( P \cdot (I_0, 0) \) is the \( n \)-dimension identity (zero) matrix; \( \text{He}(A) = A + A^T \).

**Preliminaries**

A nominal system is given as
\[
\dot{x}(t) = A(t)x(t) + A_d(t)x(t - d(t)), \quad t \geq 0,
\]
\[
x(t) = \phi(t), \quad t \in [-d_N, 0],
\]
where \( x(t) \in \mathbb{R}^n \) is state vector, \( A(t), A_d(t) \) are the real matrices with appropriate dimensions, \( d(t) \) is time-varying delay, \( \phi(t) \in C([−d_N, 0], \mathbb{R}^n) \) is the initial function.

For any \( t \geq 0 \), the time-varying delay \( d(t) \) is described as two categories:

**Case 1:** \( d(t) \) - a differentiable function satisfying.
\[
0 \leq d_0 \leq d(t) \leq d_N, \quad \mu_1 \leq \dot{d}(t) \leq \mu_2.
\]

**Case 2:** \( d(t) \) - a continuous function satisfying.
\[
0 \leq d_0 \leq d(t) \leq d_N,
\]
where \( d_0, d_N, \mu_1, \mu_2 \) are constants.

Considering that uncertainties exist in the system, parameters are represented as \( J_i(t), (i = 1, 2), \) and \( J_i(t) = A(t), J_2(t) = A_d(t), \) which are not exactly known and might be taken from an interval \( J_i(t) \in [J_{i1}, J_{i2}] \). Then the parameters with uncertainties satisfy
\[
J_i(t) = \Pi_i(t)J_{i1} + \Pi_2(t)J_{i2} = \sum_{i=1}^{2} \Pi_i(t)J_{i0},
\]
with any constant \( \Pi_i(t) \geq 0, \Pi_2(t) \geq 0 \) satisfying \( \Pi_i(t) + \Pi_2(t) = 1 \).

Some lemmas are employed as follows:

**Lemma 1** (ref. 19). Considering any matrix \( \hat{R} > 0 \), and a continuously differentiable function \( z: [\alpha, \alpha'] \rightarrow \mathbb{R}^n \), the next inequality holds
\[
-(\alpha^T - \alpha') \int_{\alpha}^{z^T} z^T(s) \alpha dz(s) \leq \varepsilon^T(t) \Theta \in (t),
\]
where
\[
\varepsilon(t) = \left[ z^T(\alpha^T)z^T(\alpha') \frac{1}{(\alpha^T - \alpha')} \int_{\alpha}^{z^T} z^T(s) dz(s) \right]^T,
\]
and
\[
\Theta = \begin{bmatrix}
-4\hat{R} & -2\hat{R} & 6\hat{R} \\
* & -4\hat{R} & 6\hat{R} \\
* & * & -12\hat{R}
\end{bmatrix}.
\]

**Lemma 2** (ref. 29). Let \( z: [\alpha, \alpha'] \rightarrow \mathbb{R}^n \) be a differentiable function, \( Z \in \mathbb{R}^{n \times n} \) and \( J_1, J_2 \in \mathbb{R}^{n \times 3n}_s \) be symmetric matrices, and \( J_2 \in \mathbb{R}^{3n \times n}_s \), \( S_1, S_2 \in \mathbb{R}^{n \times n}_s \) satisfying this condition,
\[
\begin{bmatrix}
J_1 & J_2 & S_1 \\
* & J_3 & S_2 \\
* & * & Z
\end{bmatrix} \geq 0
\]
it holds
\[
-\int_{\alpha}^{z^T} z^T(s) \underline{Z} \alpha ds \leq z^T \Pi z,
\]
where
\[
\zeta = \left[ z^T(\alpha^T)z^T(\alpha') \frac{1}{\alpha^T - \alpha'} \int_{\alpha}^{z^T} z^T(s) dz(s) \right]^T,
\]
\[
\Pi = (\alpha^T - \alpha') \left( J_1 + \frac{1}{3} J_3 \right) + \text{He}(S_1 \partial_1 + S_2 \partial_2),
\]
\[
\partial_1 = \tilde{e}_1 - \tilde{e}_2, \quad \partial_2 = 2\tilde{e}_3 - \tilde{e}_1 - \tilde{e}_2, \quad \tilde{e}_1 = [1 \ 0 \ 0],
\]
\[
\tilde{e}_2 = [0 \ 1 \ 0], \quad \tilde{e}_3 = [0 \ 0 \ 1].
\]

**Lemma 3** (ref. 30). For any vectors \( f_1, \ldots, f_N \) scalar \( \gamma(t) \in [0, 1], \sum_{i=1}^{N} f_i(t) = 1, \) and matrices \( \hat{R} > 0, \) there exists matrix \( S_i(i = 1, \ldots, N - 1, j = i + 1, \ldots, N) \) satisfies
\[
\begin{bmatrix}
R_1 & S_{ij} \\
* & R_j
\end{bmatrix} \geq 0,
\]
then the next inequality holds:
\[
-\sum_{i=1}^{N} \frac{1}{\gamma_i(t)} f_i^T R_i f_i \leq \begin{bmatrix}
\gamma_1 & \ldots & S_{iN} \\
* & & \gamma_N
\end{bmatrix} \begin{bmatrix}
f_1 \\
\ldots \\
f_N
\end{bmatrix}.
\]

**Lemma 4** (ref. 31). For a symmetric positive matrix \( \hat{R} \in \mathbb{R}^{n \times n} \) and differentiable function \( z: [\alpha, \alpha'] \rightarrow \mathbb{R}^{n \times n} \), then the next double integral inequality holds.
\[ \int_{\alpha}^{\alpha'} z^T(s) \tilde{\rho} z(s) dsd\theta \leq \eta^T(t) \Omega \eta(t), \]

where
\[ \eta(t) = \begin{bmatrix} z^T(\alpha^+), \frac{1}{(\alpha^+ - \alpha)^2} \int_{\alpha}^{\alpha'} z^T(s) ds, \ldots \end{bmatrix}^T \]

and
\[ \Omega = \begin{bmatrix} -6\tilde{\rho} & -6\tilde{\rho} & 24\tilde{\rho} \\ * & -18\tilde{\rho} & 48\tilde{\rho} \\ * & * & -144\tilde{\rho} \end{bmatrix}. \]

**Lemma 5** (ref. 32). Let \( \xi \in \mathbb{R}^n \), \( \Phi = \Phi^T \in \mathbb{R}^{n \times n} \) and \( \mathcal{B} \in \mathbb{R}^{n \times n} \) with \( \text{rank}(\mathcal{B}) < n \). The next statements are equivalent:

(i) \( \xi^T \Phi \xi < 0 \), \( \forall \mathcal{B} \xi = 0 \), \( \xi \neq 0 \);

(ii) \( \mathcal{B}^T \Phi \mathcal{B} \perp < 0 \);

(iii) \( \exists \mathcal{D} \in \mathbb{R}^{(n-\text{rank}(\mathcal{B})) \times n}, \text{He}(\mathcal{D} \mathcal{B}) < 0 \).

where
\( \mathcal{B}^T \in \mathbb{R}^{(n-\text{rank}(\mathcal{B})) \times n} \) is the right orthogonal complement of \( \mathcal{B} \).

**Stability analysis**

A GSD based delay partition method is employed in Figure 1. For any integer \( N \geq 1 \), the interval \([d_0, d_N]\) is separated into \( N \) subintervals as

\[ \rho_i = \lambda^i \]

\[ d_i = d_0 + \sum_{i=1}^{i} \frac{1}{\rho_i}, \quad i = 1, \ldots, N, \]

where \( \lambda \) is a real positive number, and \( \rho_i \) is the length of the \( i \)th subinterval that is equal to \( \lambda^i \). It is obtained as

\[ \lambda = \sqrt[k]{d_N - d_0}. \]

**Theorem 1.** Given an integer \( N > 0 \), and \( \rho_i = \lambda^i \). Consider delay \( d(t) \) satisfying Case 1. The system (1) is asymptotically stable if there exists symmetric positive definite matrices \( \hat{R}, \tilde{R}, P_j, P, Z \in \mathbb{R}^{n \times n} \) (i = 1, \ldots, N), \( \hat{P}_i \in \mathbb{R}^{n \times n} \) (i = 1, \ldots, N), \( \hat{U}_i \in \mathbb{R}^{n \times n} \) (i = k, \ldots, N), \( \mathcal{B} \in \mathbb{R}^{(N+2n)x(N+2n)}, \mathcal{F}_1, \mathcal{F}_2 \in \mathbb{R}^{n \times n} \), matrices \( \mathcal{J}_1, \mathcal{J}_2 \in \mathbb{R}^{n \times n} \), \( \mathcal{J}_3, \mathcal{J}_4 \in \mathbb{R}^{n \times n} \), \( \mathcal{G} \in \mathbb{R}^{(N+2n)x(n+2n)}, d \), such that the next LMIs hold

\[ \begin{bmatrix} \mathcal{J}_1 & \mathcal{J}_2 & \mathcal{S}_1 \\ \ast & \mathcal{J}_2 & \mathcal{S}_2 \\ \ast & \ast & \tilde{\mathcal{R}} \end{bmatrix} \geq 0, \quad i = 1, \ldots, N, \]

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{G}_{ij}(t) (\Xi_{ij} + \text{He}(\mathcal{G}_{ij})) < 0, \quad \mathcal{G}_{ij}(t) = \sum_{k=1}^{K} \mathcal{G}_{ij}(t) \mathcal{G}_{ij}(t), \quad k = 1, \ldots, N, \]

**Figure 1.** GSD delay partition method.
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where
\[ \mathcal{X}_0 = A(t) e_1^T + A_y(t) e_{N+4}^T - e_{N+3}^T \]
\[ \Xi_{k, k_2} = \Xi_{1, k_2} + \Xi_{2, k_2} + \Xi_{3, k_2} + \Xi_{4, k_2} + \Xi_{5, k_2} + e_{N+4j} \hat{e}_{N+4j}^T \]
\[ \Xi_{4, k} = \left[ \begin{array}{c} e_i^T \\ e_{N+5}^T \\ \vdots \\ e_{2N+4}^T \\ e_{2N+5}^T \end{array} \right]^T \begin{bmatrix} \mathbf{B} \end{bmatrix} = \left[ \begin{array}{c} e_i^T \\ e_{N+5}^T \\ \vdots \\ e_{2N+4}^T \\ e_{2N+5}^T \end{array} \right]^T \left[ \begin{bmatrix} e_i^T \\ e_{N+5}^T \\ \vdots \\ e_{2N+4}^T \\ e_{2N+5}^T \end{bmatrix} \right] \]
\[ \Xi_{4, k} = \sum_{i=1}^{N} \left( e_{i+1}^T e_{i+1}^T - e_{i+2}^T e_{i+2}^T \right) + e_{k+1}^T \hat{e}_{k+1}^T \]
\[ \Xi_{3, k_2} = \sum_{i=1}^{N} \left( e_{i+1}^T \tilde{P}_i e_{i+1}^T - e_{i+2}^T \tilde{P}_i e_{i+2}^T \right) + e_{k+1}^T \tilde{P}_k e_{k+1}^T \]
\[ \Lambda_1 = d^2 \left( f_j^1 + \frac{1}{3} f_j^2 \right) + d_1 \mathbf{H} e \left( S_1 \theta_1 + S_2 \theta_2 \right) \]
\[ \Lambda_2 = \left[ \begin{array}{ccc} -4 \hat{R}_1 & -2 \hat{R}_1 & 6 \hat{R}_1 \\ 0 & -4 \hat{R}_1 & 6 \hat{R}_1 \\ 0 & 0 & -12 \hat{R}_1 \end{array} \right] \]
\[ \Lambda_3 = \left[ \begin{array}{ccc} \hat{R}_k & -\hat{R}_k & J \\ 2 \hat{R}_k & -J & -J \end{array} \right] \]
\[ \Lambda_4 = \left[ \begin{array}{ccc} -6 \hat{Z}_1 & -6 \hat{Z}_1 & 24 \hat{Z}_1 \\ 0 & -18 \hat{Z}_1 & 48 \hat{Z}_1 \\ 0 & 0 & -144 \hat{Z}_1 \end{array} \right] \]

Proof. For \( \forall t \geq 0 \), there exists an integer \( k \in \{1, \ldots, N\} \), such that \( d(t) \in l_i \). The LKF is proposed as
\[ \mathcal{V}(x_i, k) = \mathcal{V}_1(x_i) + \mathcal{V}_2(x_i) \]
where
\[ \mathcal{V}_1(x_i) = \int_{t-d(t)}^{t} x(t) \phi_x(t) \left[ \begin{array}{c} x(t) \\ \phi_x(t) \end{array} \right] \]
\[ \mathcal{V}_2(x_i) = \sum_{i=1}^{N} \int_{t-d(t)}^{t} x(t) \phi_x(t) ds \]
with
\[ \mathcal{V}_3(x_i, k) = \sum_{i=1}^{k-1} \int_{t-d(t)}^{t} x(t) \phi_x(t) ds \]
The derivative of $V_2(x_i)$ is derived as
\[
V_2'(x_i) = \sum_{i=1}^{N} (x^T (t-d_{i-1}) P_i x(t-d_{i-1}))
- x^T (t-d_i) P_i x(t-d_i) + x^T (t) \tilde{P}_i x(t)
- \left( 1 - \sum_{i=1}^{N} \sigma_i(t) \mu_i \right) x^T (t-d(t)) \hat{P}_i x(t-d(t))
= \phi^T (t) \sum_{i=1}^{2} \sigma_i(t) \Xi_{2i} \phi(t).
\] (15)

$V_1'(x_i, k)$ is derived as
\[
V_1'(x_i, k) = \sum_{i=1}^{N} (x^T (t-d_{i-1}) \hat{P}_i x(t-d_{i-1}))
- x^T (t-d_i) \hat{P}_i x(t-d_i)
+ x^T (t-d_{k-1}) \hat{P}_i x(t-d_{k-1})
- \left( 1 - \sum_{i=1}^{N} \sigma_i(t) \mu_i \right) x^T (t-d(t)) \hat{P}_i x(t-d(t))
+ \left( 1 - \sum_{i=1}^{N} \sigma_i(t) \mu_i \right) x^T (t-d(t)) \hat{P}_i x(t-d(t))
- x^T (t-d_k) \hat{U}_i x(t-d_k)
+ \sum_{i=1}^{N} (x^T (t-d_{i-1}) \hat{U}_i x(t-d_{i-1}))
- x^T (t-d_i) \hat{U}_i x(t-d_i)
= \phi^T (t) \sum_{i=1}^{N} \sigma_i(t) \Xi_{3i} \phi(t).
\] (16)

The derivative of $V_3(x_i, k)$ is deduced as
\[
V_3'(x_i, k) = \hat{x}^T \left( \sum_{i=1}^{N} \sigma_i(t) \mu_i \right) \hat{R}_i \hat{x}(t)
- \sum_{i=1}^{N} (d_0 + \sum_{i=1}^{N} \hat{R}_i \hat{x}(s)ds)
- \sum_{i=1}^{N} \left( \sum_{i=1}^{N} \sigma_i(t) \mu_i \right) \hat{x}^T (s) \hat{R}_i \hat{x}(ds).
\] (17)
Applying Lemma 2 to deal with the second term of (17), it is given as
\[ -\sum_{i=1}^{N} \left( d_0 + \sum_{a=1}^{i} \lambda^a \right) \int_{t}^{t-d_i} \hat{x}^T(s) \hat{R}_i \hat{x}(s) ds \]
\[ \leq \sum_{i=1}^{N} \theta_i(t) A_i \theta_i(t), \quad (18) \]
where
\[ \theta_i(t) = \left[ \hat{x}^T(t) (t-d_i) \frac{1}{d_i} \int_{t-d_i}^{t} \hat{x}^T(s) ds \right]^T. \]

A_i is defined in Theorem 1.

Using Lemma 1 and 3 for the third term of (17), it is derived as
\[ -\sum_{i=1}^{N} \lambda^i \int_{t-d_i}^{t-(d_i+\sum_{a=1}^{i} \lambda^a)} \hat{x}^T(s) \hat{R}_i \hat{x}(s) ds \]
\[ = - \sum_{i=1, i \neq k}^{N} \lambda^i \int_{t-d_i}^{t-(d_i + \sum_{a=1}^{i} \lambda^a)} \hat{x}^T(s) \hat{R}_i \hat{x}(s) ds \]
\[ - \rho k \int_{t-d_i}^{t-d_k} \hat{x}^T(s) \hat{R}_k \hat{x}(s) ds \]
\[ \leq \sum_{i=1, i \neq k}^{N} \theta_i(t) A_2 \theta_i(t) - \theta_k(t) A_3 \theta_k(t), \quad (19) \]
where
\[ \theta_i(t) = \left[ \hat{x}^T(t-d_i) \frac{1}{d_i} \int_{t-d_i}^{t} \hat{x}^T(s) ds \right]^T, \]
\[ \theta_k(t) = \left[ \hat{x}^T(t-d_k) \frac{1}{d_k} \int_{t-d_k}^{t} \hat{x}^T(s) ds \right]^T. \]

A_2 and A_3 are defined in Theorem 1. Then, it follows from eqs (17)–(19) that
\[ \dot{V}_1(x(t)) \leq \dot{x}^T(t) \left( \sum_{i=1}^{N} \lambda^i \hat{R}_i + \sum_{i=1}^{N} \left( d_0 + \sum_{a=1}^{i} \lambda^a \right) \hat{R}_i \right) \dot{x}(t) \]
\[ + \phi^T(t) \Xi \phi(t). \quad (20) \]

Applying Lemma 4, the above last two terms are derived as
\[ -\int_{-d_i}^{t} \hat{x}^T(s) \hat{Z}_i \hat{x}(s) ds \]
\[ - \sum_{i=2}^{N} \theta_i(t) A_4 \theta_i(t), \quad (21) \]

where
\[ \theta_i(t) = \left[ \hat{x}^T(t-d_i) \frac{1}{d_i} \int_{t-d_i}^{t} \hat{x}^T(s) ds \right]^T, \]
\[ \frac{1}{(\rho t)^T} \int_{-d_i}^{t} \hat{x}^T(s) ds \theta_i(t). \quad (22) \]

A_4 is defined in Theorem 1. Then it is derived as
\[ \dot{V}_1(x(t)) \leq \dot{x}^T(t) \left( \sum_{i=1}^{N} \lambda^i \hat{R}_i + \sum_{i=1}^{N} \left( d_0 + \sum_{a=1}^{i} \lambda^a \right) \hat{R}_i \right) \dot{x}(t) \]
\[ + \phi^T(t) \Xi \phi(t). \quad (23) \]
Hence, the next inequality holds

$$
\mathcal{V}(x, t) \mid_{d(t) \leq \xi} \leq \varphi^T(t) \sum_{s=1}^2 \sigma_{s, 1} \sum_{s=1}^2 \sigma_s(t) \Xi_{ks, s_2} \varphi(t).
$$

(24)

Applying the vector (eq. 10) with the simplified expression (eq. 9), the linear system (eq. 1) is represented as

$$
0 = \tilde{\mathbf{g}}_0 \varphi(t),
$$

(25)

where \( \tilde{\mathbf{g}}_0 \) is described in Theorem 1.

Hence, the asymptotic stability result of the system (eq. 1) is presented as

$$
\varphi^T(t) \sum_{s=1}^2 \sigma_{s, 1} \sum_{s=1}^2 \sigma_s(t) \Xi_{ks, s_2} \varphi(t) < 0
$$

subject to: $0 = \tilde{\mathbf{g}}_0 \varphi(t).$

(26)

So, using Lemma 5, there will exist a matrix \( \mathbf{G} \) such that eq. (26) is equivalent to

$$
\varphi^T(t) \sum_{s=1}^2 \Pi_0(t) \sum_{s=1}^2 \sigma_{s, 1} \sum_{s=1}^2 \sigma_s(t) (\Xi_{ks, s_2} + \text{He}(\tilde{\mathbf{G}}_{0})) \varphi(t) < 0.
$$

(27)

Thus the newly proposed LKF derivative is obtained as \( \mathcal{V}(x, k) \mid_{d(t) \leq \xi} < 0. \) This means \( \mathcal{V}(x, k) \mid_{d(t) \leq \xi} \leq \xi \| x(t) \|^2 \) for a sufficiently small \( \xi > 0. \) So the system (eq. 1) is asymptotically stable. This completes the proof.

**Theorem 2.** Given an integer \( N \geq 0 \), and \( \rho_1 = \lambda' \). Consider delay \( d(t) \) satisfying Case 2. The system (eq. 1) is asymptotically stable if there exists symmetric positive definite matrices \( \tilde{\mathbf{R}}, \tilde{\mathbf{K}}, \mathbf{P}_1, \mathbf{P}_2 \in \mathbb{R}^{n \times n} \) with \( \mathbf{R} \in \mathbb{R}^{(N+1) \times (N+1)} \) and \( \mathbf{J}_1, \mathbf{J}_2 \in \mathbb{R}^{3 \times 3 \times n} \), symmetric matrices \( \mathbf{S}_1, \mathbf{S}_2 \in \mathbb{R}^{3 \times 3 \times n} \), and \( \mathbf{F} \in \mathbb{R}^{(N+5) \times n \times n} \), such that the next LMI holds

$$
\begin{bmatrix}
    \mathbf{J}_1 & \mathbf{J}_2 & \mathbf{S}_1 \\
    * & \mathbf{J}_3 & \mathbf{S}_2 \\
    * & * & \tilde{\mathbf{R}}
\end{bmatrix} \geq 0, \quad i = 1, \ldots, N.
$$

(28)

$$
\sum_{o=1}^N \Pi_0(t) \tilde{\Xi}_k + \text{He}(\tilde{\mathbf{G}}_{0}) < 0.
$$

(29)

where \( \Xi_{ks, s_2} \) is modified to \( \tilde{\Xi}_k \) by removing \( \Xi_{3k_1, s_2} \) and replacing \( \Xi_{3k_1, s_2} \) to \( \Xi_{1, s_2} \) as follows

$$
\tilde{\Xi}_k = \tilde{\Xi}_1 + \tilde{\Xi}_2 + \Xi_{4k_1} + \Xi_{5} + \epsilon_{N+3} \Xi_{N+3}^T.
$$

$$
\tilde{\Xi}_1 = \text{He} \left( \begin{bmatrix}
    e_{1}^T_1 & e_{1}^T_{N+5} \\
    e_{1}^T_{N+5} & \mathbf{R} \\
    \vdots & \vdots \\
    e_{2N+4}^T & e_{2N+4}^T
\end{bmatrix} \right).
$$

\( \tilde{\mathbf{g}}_o \) are defined in Theorem 1.

**Proof.** Modify the Lyapunov functionals (eq. (13)) by changing \( \mathcal{V}_1(x, k) \) and removing \( \mathcal{V}_3(x, k) \) as

$$
\mathcal{V}(x, k) \mid_{d(t) \leq \xi} = \mathcal{V}_1(x, k) + \mathcal{V}_2(x, k) + \mathcal{V}_3(x, k),
$$

(30)

where

$$
\varphi_1^T \left( \begin{array}{c}
    x(t) \\
    \varphi_2(t)
\end{array} \right)^T \mathbf{B} \left( \begin{array}{c}
    x(t) \\
    \varphi_2(t)
\end{array} \right),
$$

$$
\mathcal{V}_2(x, k) = \sum_{i=1}^N \int_{t-i}^{t} x^T(s-d_{i-1}) \mathbf{P}_i x(s-d_{i-1}) ds.
$$

Then following the same procedures of proof of Theorem 1, the stability criteria will be equivalent to

$$
\varphi^T(t) \sum_{s=1}^2 \Pi_0(t) \tilde{\Xi}_k + \text{He}(\tilde{\mathbf{G}}_{0}) \varphi(t) < 0.
$$

(31)

This completes the proof.

**Illustrative examples**

**Example 1.** Consider a nominal system (eq. (1)) with the parameters discussed in refs 28 and 34 are given as

$$
A = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix}.
$$

Considering \( \pm 20\% \) parameters uncertainties regarding eq. (4), i.e. \( \mathbf{J}_1(t) = [\mathbf{J}_1(t) \times 80\% , \mathbf{J}_1(t) \times 120\%] \). By choosing \( d_0 = 0 \) and the maximum value of \( d_N \) is obtained with different \( \mu \).

In the case of \( \mu = 0.9 \) and \( d_N = 0.5412 \), the state response of eq. (1) is shown in Figure 2.

Table 1 compares maximum upper bounds \( d_N \) with different values of \( \mu \) and \( d_0 = 0 \). It clearly shows that for \( \mu = 0.9 \) and \( \mu \geq 1 \) this proposed approach presents a bigger \( d_N \) than the results in refs 28 and 34, with \( N = 3 \). Figure 2 indicates that the state converges to zero with \( d_N = 0.5412 \), which means the system (eq. (1)) is globally asymptotically stable under the obtained maximum value of \( d(t) \).

**Example 2.** Consider a nominal system (eq. (1)) with the parameters discussed in refs 31 and 33, are given as

$$
A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.
$$

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Considering time-varying delay satisfying case 2, the maximum value of $d_N$ is obtained based on Theorem 2.

In Table 2, considering unknown $\mu$ and different $d_0$, the maximum upper bounds of $d_N$ are obtained. It shows that the proposed method improves the stability results compared to the previous works\textsuperscript{28,31,33}.

**Remark 1.** In this article, parameter uncertainties are taken into account for stability analysis of delayed linear system. Different from the existing results\textsuperscript{28,31,33,34}, a new LKF is constructed, which contains GSD-dependent integral forms with unfixed intervals. This novel design can reduce the number of decision variables and provide expected stability conditions with high efficiency and less computational complexity. A new expression of uncertainties is formulated in (eq. (4)) using convex combination method. Extra inequalities and constraint conditions can be eliminated compared to earlier research\textsuperscript{14,21,31}. The stability criteria on robust analysis for linear system with time-varying delay is improved.

### Table 1. Maximum value of $d_N$ with $d_0 = 0$ and different values $\mu$

<table>
<thead>
<tr>
<th>Method</th>
<th>Peng and Tian\textsuperscript{34}</th>
<th>Liu and Li\textsuperscript{28}</th>
<th>Theorem 1 ($N = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 0.9$</td>
<td>0.4760</td>
<td>0.5390</td>
<td>0.5412</td>
</tr>
<tr>
<td>$\mu \geq 1$</td>
<td>0.4670</td>
<td>0.5390</td>
<td>0.5407</td>
</tr>
</tbody>
</table>

### Table 2. Upper bounds of $d_N$ for unknown $\mu$ and different $d_0$

<table>
<thead>
<tr>
<th>Method</th>
<th>Lee et al.\textsuperscript{15}</th>
<th>Park et al.\textsuperscript{31}</th>
<th>Theorem 2 ($N = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_0 = 0$</td>
<td>1.35</td>
<td>1.64</td>
<td>1.71</td>
</tr>
<tr>
<td>$d_0 = 1$</td>
<td>2.31</td>
<td>2.91</td>
<td>3.13</td>
</tr>
</tbody>
</table>

**Remark 2.** This employed GSD method considerably improves the efficiency for obtaining the maximum value of $d(t)$. Sum formulation of the GSD method is $a_i(1 - \lambda^2)(1 - \lambda)$ where $a_i$ is the first term and $\lambda \neq 1$. For example, let the first term $a_1 = 1$, $\lambda = 2$ and the partitioning number $N = 4$, that is $a_i(1 - \lambda^2)(1 - \lambda) = 15$. This means that the partition number $N = 15$, if the common equivalent division method is selected. However, by using our GSD approach, the partition number $N = 4$, which is 30% less than the equivalently partitioning method. Hence, the decision variables are reduced considerably. In addition, if the common ratio $\lambda = 1$, then the sum will be $N \times a_1$ that covers the length of subinterval to equal. Thus the previously produced works\textsuperscript{14,35} using equivalent partition approach are the special cases of this proposed method.

**Remark 3.** When a system has high dimension, the computation burden is increased. It becomes more difficult to work out feasible solution. Lower dimension systems are commonly used for stability study. Additionally, big partitioning number requires much more computing time. In future, we will try to discover an improved method to reduce the computation cost.

### Conclusion

In this study, stability conditions of nominal system with parameter uncertainties and interval time-varying delay are investigated by utilizing the GSD delay-partitioning method and convex combination approach. New GSD-dependent LKF is developed, which includes integral forms with geometric progression interval. Additionally, the convex combination method is proposed to flexibly estimate LKF derivative instead of using extra inequalities. Therefore, unexpected enlargement can be appropriately reduced. Meanwhile, less decision variables are used, because the proposed GSD approach reduces the number of partitioning subintervals. As a result, the computational burden is lessened. Numerical results demonstrate a good stability criteria. Due to the complex dynamics of nonlinear system, control of such systems is full of challenges. Recently, this research area has attracted a lot of attention. Thus, future studies should carry on the robust control of nonlinear systems with stochastic disturbances and uncertainties.

Conflict of interest. All authors declare that there are no conflict of interests regarding the contents presented in this article.


