Uncertainty trade-off and disturbance trade-off for quantum measurements

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An important non-classical feature of quantum measurements is the celebrated uncertainty trade-off, namely that the uncertainties in the outcomes of measurements performed on *distinct* yet *identically prepared* ensembles of systems cannot *all* be made arbitrarily small. Recently, we have shown that quantum measurements also exhibit another non-classical feature of disturbance trade-off namely, that the disturbances associated with measurements performed on *distinct* yet *identically* prepared ensembles of systems in a pure state cannot *all* be made arbitrarily small. In this article, we review the known results on uncertainty trade-off and disturbance trade-off for projective and non-projective measurements.

Keywords: Disturbance, entropy, projective measurement, uncertainty.

Introduction

THE uncertainty principle is one of the central distinguishing features of quantum mechanics, and plays an important role in the study of quantum information theory and quantum cryptography. In its original formulation envisioned by Heisenberg[1], the uncertainty principle was stated as an effect of the *disturbance* caused due to a measurement of one observable on a succeeding measurement of another. However, the subsequent mathematical formulation due to Robertson[2] and Schrödinger[3] in terms of variances, and the more recent entropic formulations of the uncertainty principle[4], pertain to an entirely different situation. They demonstrate the existence of a fundamental trade-off for the uncertainties associated with *independent* measurements of incompatible observables on *identically prepared* ensembles of systems.

In this article, we review our recent results on the existence of a similar principle of trade-off for the *disturbances* associated with the measurements of a set of observables[5]. It is a fundamental feature of quantum theory that when an observable is measured on an ensemble of systems, the density operator of the resulting ensemble is in general different from that prior to the measurement. The distance between these two density operators is therefore a measure of the *disturbance* due to measurement. Different measures of distance between density operators[6,7] give rise to different measures of disturbance. For a general class of such disturbance measures, we demonstrate the existence of a fundamental trade-off principle for the disturbances associated with quantum measurements performed on *distinct* yet *identically prepared* copies of a pure state.

We show that the average of the disturbances associated with a set of projective measurements is strictly greater than zero whenever the associated set of observables do not have a common eigenvector. In the particular case when the disturbance is characterized by the square of the fidelity function, there is a mathematical equivalence between the disturbance due to the measurement of an observable on a pure state and the uncertainty as quantified by the Tsallis entropy (T_2) of order 2 of the probability distribution over the outcomes of such a measurement. This provides a new operational significance to the T_2 entropy in the context of quantum information theory. Some of the known results on entropic uncertainty relations (EURs) can be made use of to obtain disturbance trade-off relations for specific classes of observables. We also show an optimal disturbance trade-off relation for a pair of qubit observables, which is based on a new, tight T_2 EUR.

For the more general class of observables given by positive operator valued measures (POVMs), the associated measurements are characterized by completely positive (CP) instruments. For this class of non-projective measurements, we show that the disturbance and uncertainty trade-offs are significantly different; they indeed capture different aspects of the complementarity of a set of measurements.

The rest of the article is organized as follows. We begin with a brief review of the mathematical formalism of uncertainty trade-offs in the form of entropic uncertainty relations. Next, we define the class of disturbance measures, derive the trade-off principle for projective measurements and discuss the equivalence between the fidelity-based measure and the Tsallis entropy of order 2 (T_2). Finally, we discuss the disturbance trade-off principle for non-projective measurements.

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Uncertainty trade-off for quantum measurements

We restrict our attention to observables and measurements with a discrete set of outcomes. In conventional quantum mechanics, these correspond to self-adjoint operators with a purely discrete spectrum. Any such observable A has a spectral resolution

$$A = \sum_i a_i P_i^A,$$

where $\{P_i^A\}$ is a discrete projection valued measure (PVM), that is, $\{P_i^A\}$ are projectors satisfying

$$P_i^A P_j^A = P_i^A \delta_{ij}, \quad \sum_i P_i^A = I.$$

If ρ is the density operator representing the state of an ensemble of systems, the probabilities for obtaining various outcomes are given by $p_i = \mathrm{tr}[\rho P_i^A]$.

More generally, a quantum observable with discrete outcomes is represented by a POVM. A POVM $\mathcal{A}$ is characterized by a set of positive operators $\{A_i\}$ satisfying $0 \le A_i \le \mathbb{I}$, with $\sum_i A_i = \mathbb{I}$. The probabilities of obtaining various outcomes are now given by $p_i = \mathrm{tr}[\rho A_i]$.

Entropic measures of uncertainty

The *uncertainty* in the outcome of a measurement is reflected in the *spread* of the associated probability distribution. In information theory, this is measured by various measures of entropy of the probability distribution. For example, we have the Shannon entropy, defined as $H(\{p_i\}) = -\sum_i p_i \log p_i$. More generally, we have the Rényi class of entropies

$$H_\alpha(\{p_i\}) := \frac{1}{1-\alpha} \log\left(\sum_i (p_i)^\alpha \right),$$

of which the Shannon entropy is a special case, obtained as $\lim_{\alpha\to 1}, H_\alpha$. For a finite number of outcomes d, Shannon and Rényi entropies satisfy $0 \le H_\alpha \le \log d\ (\forall \alpha)$, where the lower limit is attained for the deterministic or *zero-spread* probability distribution $p_i = \delta_{ij}$, for some j and the upper limit is attained for the *uniform* distribution $p_i = 1/d, \forall i$.

We also have the Tsallis class of entropies defined as follows[8]

$$T_\beta(\{p_i\}) := \frac{1}{1-\beta}\left(\sum_i p_i^\beta - 1 \right).$$

The Tsallis entropy reduces to the Shannon entropy in the limit $\beta \to 1$. In particular

$$0 \le T_2(\{p_i\}) = 1 - \sum_i (p_i)^2 \le 1 - \frac{1}{d},$$

where once again the lower limit is attained for the *zero-spread* probability distribution $p_i = \delta_{ij}$ for some j and the upper limit is attained for the *uniform* distribution $p_i = 1/d, \forall i$.

Uncertainty trade-off

The *uncertainty* in the outcome of a measurement of observable $A = \sum_i a_i P_i^A$ in state ρ is given by $S(A;\rho) = S(\{\mathrm{tr}[\rho P_i^A]\})$, where S is any entropic measure. In the case of Shannon and Tsallis entropies, and the Rényi entropy for $\alpha \le 1$, $S(A;\rho)$ is a *concave* function of ρ. Thus, for a pair of observables A, B, an entropic uncertainty relation (EUR) is a state-independent lower bound on the average of the entropies of A, B, of the form

$$\frac{1}{2}[S(A;\rho) + S(B;\rho)] \ge C_S(A,B), \quad \forall \rho,$$

where the observables A and B are measured on *distinct* and *identically prepared* ensembles of systems. Entropic uncertainty relations have been obtained for specific classes of observables for both the Shannon and Rényi entropies[9–17], as well as for the Tsallis entropies[18–21]. For a comprehensive survey on EURs, we refer to the review article by Wehner and Winter[4]. EURs play a central role in proving security of quantum cryptographic protocols and are often thought to provide a measure of incompatibility of quantum measurements.

For the case of projective measurements, we recall the well-known necessary and sufficient condition for zero uncertainty trade-off.

Lemma 1:

$$C_S(A,B) = \inf_\rho \frac{1}{2}[S(A;\rho) + S(B;\rho)] = 0,$$

iff the observables A, B have a common eigenvector.

For a POVM $\mathcal{A}$, the associated uncertainty can be similarly defined in terms of the entropy $S(\mathcal{A},\rho) = S(\{\mathrm{tr}[\rho A_i]\})$. The well-known Shannon EUR for a pair of POVMs $\mathcal{A}, \mathcal{B}$ derived by Krishna and Parthasarathy[22] states that

$$[S(\mathcal{A};\rho) + S(\mathcal{B};\rho)] \ge -\log \sup_{i,j} \| A^{1/2}; B_j^{1/2} \|, \forall \rho. \tag{1}$$

The following lemma[5] gives the condition under which the uncertainty trade-off vanishes for a pair of POVMs.

Lemma 2: *Let $\mathcal{A} \sim \{A_i\}$ and $\mathcal{B} \sim \{B_i\}$ be two POVMs. If S is any suitable entropy measure, the lower bound on the average uncertainty*

$$c_S(\mathcal{A},\mathcal{B}) = \inf_\rho \frac{1}{2}[S(\mathcal{A};\rho) + S(\mathcal{B};\rho)] = 0,$$

iff there exists a state $|\psi\rangle$ such that $A_k|\psi\rangle = B_l|\psi\rangle = |\psi\rangle$ for some k, l.

Note that the Shannon EUR lower bound for POVMs stated in eq. (1) is consistent with this condition.

Projective and non-projective measurements

For an observable with a discrete spectrum, the *von Neumann-Lüders collapse postulate* specifies the state of a system after the measurement. When an ensemble of systems in state ρ is subject to a measurement of the observable $A = \sum_i a_i P_i^A$, the post-measurement state of the sub-ensemble of all those systems which yield outcome a_i is given by

$$P_i^A \rho P_i^A / \mathrm{tr}[\rho P_i^A].$$

Thus, the post-measurement state of the entire ensemble is given by

$$\Phi^A(\rho) = \sum_i P_i^A \rho P_i^A. \tag{2}$$

Uncertainty trade-offs for sequential projective measurements

Once we have the collapse postulate given in eq. (2), we can discuss the uncertainty trade-off between observables A, B which are measured *sequentially* on a system in state ρ. Now, the uncertainty in the outcome of an A-measurement is given as before by $S(A; \rho)$, but the uncertainty of the outcome of the subsequent B-measurement is given by $S(B; \Phi^A(\rho))$. The following is the optimal Shannon EUR for sequential measurement of a pair of observables[23].

Theorem 3: *The optimal Shannon EUR for a pair of observables A, B with discrete spectra measured sequentially on state ρ is given by:*

$$\inf_\rho [S(A;\rho) + S(B;\Phi^A(\rho))]$$
$$= \inf_i \inf_{P_i^A|\psi\rangle = |\psi\rangle} -\sum_j \langle\psi | P_j^B | \psi\rangle \log \langle\psi | P_j^B | \psi\rangle. \tag{3}$$

Further, when A and B have non-degenerate spectra,

$$\inf_\rho [S(A;\rho) + S(B;\Phi^A(\rho))]$$
$$= \inf_i - \sum_j |\langle a_i | b_j\rangle|^2 \log |\langle a_i | b_j\rangle|^2 . \tag{4}$$

Thus, in the sequential measurement case also, the uncertainty trade-off bound vanishes iff the two observables have a common eigenvector. Further, it can also be shown that the uncertainty trade-off for sequential measurements is greater than or equal to that for distinct measurements done on identically prepared systems

$$\inf_\rho [S(A;\rho) + S(B;\Phi^A(\rho))]$$
$$\geq \inf_\rho [S(A;\rho) + S(B;\rho)]. \tag{5}$$

In Figure 1, we compare the different entropic uncertainty lower bounds for a pair of qubit observables with Bloch vectors at an angle θ. $\Lambda_{D_1}(\theta)$ is the Deutch bound, $\Lambda_{D_2}(\theta)$ is the Maassen–Uffink bound, and $\Lambda_D(\theta)$ is the optimal bound for the case of *distinct* measurements due to Sanchez-Ruiz[15] and Ghirardi *et al.*[24]. $\Lambda_S(\theta)$ is the optimal bound for the *successive* measurement scenario given in eq. (4).

Non-projective measurements

For a POVM $\mathcal{A} \sim \{A_i\}$, there is no canonical specification of the post-measurement state; the associated measurement transformation can now be chosen as any CP instrument $\Phi^\mathcal{A}$ implementing the POVM $\mathcal{A}$ (ref. 25). A CP instrument $\Phi^\mathcal{A}$ implementing $\mathcal{A}$ is a collection of completely positive linear maps $\Phi_i^\mathcal{A}$ such that, the probability of realizing outcome i is given by

$$\mathrm{tr}[\Phi_i^\mathcal{A}(\rho)] = \mathrm{tr}[\rho A_i], \quad \forall \rho.$$

The overall transformation of state ρ by instrument Φ^A is described by a *quantum channel*, that is, a completely positive trace-preserving (CPTP) map (also denoted by $\Phi^\mathcal{A}$)

$$\Phi^\mathcal{A}(\rho) = \sum_i \Phi_i^\mathcal{A}(\rho).$$

Recall that *any* CPTP map can be represented in the form $\Phi^\mathcal{A}(\rho) = \sum_i K_i \rho K_i^\dagger$, where the Kraus operators $\{K_i\}$ satisfy $\sum_i K_i^\dagger K_i = \mathbb{I}$. The same observable can indeed be implemented by several different instruments. One simple implementation of a measurement of observable $\mathcal{A} \sim \{A_i\}$

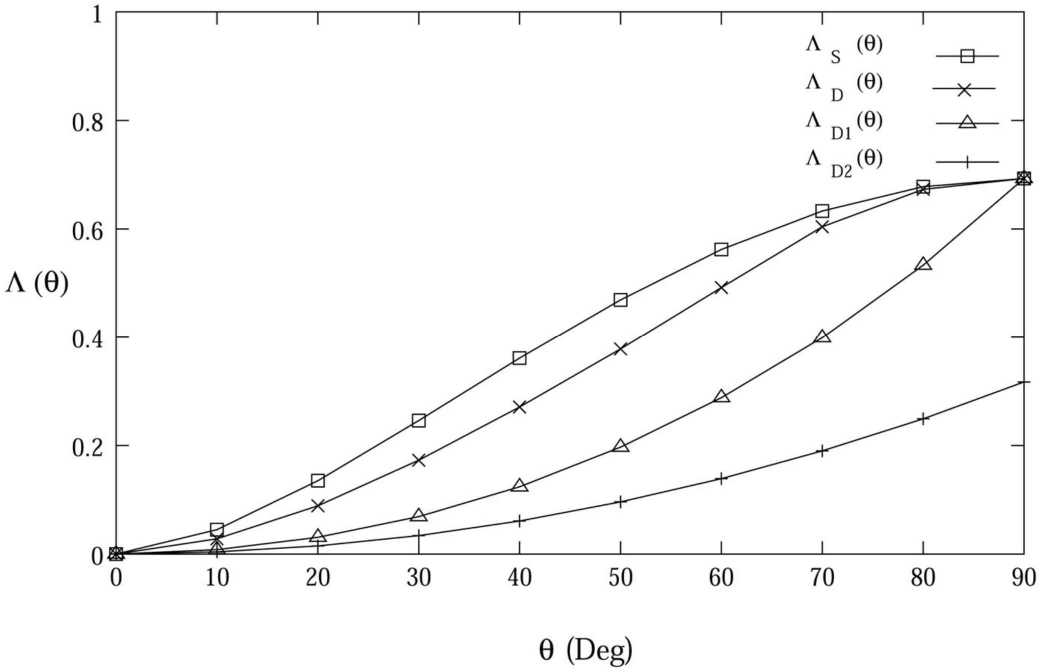

Figure 1. Uncertainty bounds for a pair of qubit observables with Bloch vectors at an angle θ (reproduced from ref. 23).

is given by the Lüders instrument $\Phi^{A}_{\mathcal{L}}$, in which the post-measurement state after a measurement of observable $\mathcal{A}$ on state ρ is given by

$$\Phi^{A}_{\mathcal{L}}(\rho) = \sum_i A_i^{1/2} \rho A_i^{1/2}.$$

The class of CP instruments is the appropriate generalization of the von Neumann–Lüders collapse postulate, viewed as a transformation on the class of density operators. Unfortunately, it does not provide any appropriate generalization of the von Neumann–Lüders collapse postulate for observables with a continuous spectrum, namely a generalization which satisfies either the *generalized Born statistical formula* (which is the standard prescription for joint probabilities of commuting observables)[26,27] or the *repeatability property*[28].

This can be seen by noting that the *dual* of the measurement transformation $\rho \to \Phi^{A}(\rho) = \sum_i P_i^A \rho P_i^A$, namely the map $C \to \sum_i P_i^A C P_i^A$ defines a *conditional expectation* on the set of all bounded operators $\mathcal{B}(\mathcal{H})$. The map $C \to \sum_i P_i^A C P_i^A$ characterizes a normal conditional expectation (*a la* Umegaki, Nakamura, Turumuru, Tomiyama[29]), $\mathcal{B}(\mathcal{H}) \to \mathcal{U}'_A$, where $\mathcal{U}'_A$ is the commutant $\{P_i^A\}'$, i.e. the set of all bounded operators commuting with A. The generalized Born statistical formula implies that the dual of the CPTP channel that characterizes the collapse should indeed be such a normal conditional expectation. However, it is a general mathematical result due to Arveson, Stormer[30] and Davies[31] that there are no such normal conditional expectations onto the commutant generated by the projectors of an observable with a continuous spectrum. Hence, there is no CPTP instrument implementing the measurement of an observable with continuous spectrum, which is consistent with the generalized Born statistical formula.

Measures of disturbance

As explained above, for a general quantum measurement $\mathcal{A}$ on an ensemble of systems in state ρ, the postmeasurement state $\Phi^{\mathcal{A}}(\rho)$ of the ensemble is described via the action of a CPTP map $\Phi^{\mathcal{A}}$. The *distance* between the states ρ and $\Phi^{\mathcal{A}}(\rho)$ is a valid measure of the *disturbance* caused to state ρ by a measurement $\mathcal{A}$.

Using some of the standard measures of distance between density operators[6], we define the following measures of disturbance due to measurement $\mathcal{A}$

$$\mathfrak{D}_1(\mathcal{A};\rho) \equiv \frac{1}{2}\mathrm{tr}\,|\,\Phi^{\mathcal{A}}(\rho) - \rho\,|,$$

$$\mathfrak{D}_F(\mathcal{A};\rho) \equiv 1 - F^2[\Phi^{\mathcal{A}}(\rho), \rho],$$

$$\mathfrak{D}_\infty(\mathcal{A};\rho) \equiv \|\,\Phi^{\mathcal{A}}(\rho) - \rho\,\|, \tag{6}$$

where $\mathrm{tr}|C| = \mathrm{tr}(C^\dagger C)^{1/2}$ is the trace-norm, $F[\rho,\sigma] = \mathrm{tr}[\sigma^{1/2}\rho\sigma^{1/2}]^{1/2}$ is the fidelity and $\|C\|$ is the operator norm.

All three disturbance measures satisfy

$$0 \le \mathfrak{D}_\alpha(\mathcal{A};\rho) \le 1, \quad \alpha \in \{1, F, \infty\},$$

with $\mathfrak{D}_\alpha(\mathcal{A};\,\rho) = 0$ iff $\Phi^{\mathcal{A}}(\rho) = \rho$. Recently, this class of disturbance measures has been used in the context of quantifying *incompatibility* of a pair of observables[7]. The distance $\frac{1}{2}\mathrm{tr}\,|\,\Phi^{\mathcal{A}}(\rho)-\rho\,|$ is *convex* in ρ, so that the corresponding disturbance measure $\mathfrak{D}_1(\mathcal{A};\,\rho)$ attains its supremum for pure states. Similarly, the measures $\mathfrak{D}_F(\mathcal{A};\,\rho)$ and $\mathfrak{D}_\infty(\mathcal{A};\,\rho)$ also attain supremum for pure states.

The following lemma[5] summarizes the conditions for a pure state to be left *undisturbed* by a measurement.

Lemma 4 (Zero-disturbance conditions): *(a) For the projective measurement associated with a self-adjoint operator A with a purely discrete spectrum,* $\mathfrak{D}_\alpha(\mathcal{A};\,|\psi\rangle) = 0$ ($\alpha \in \{1, F, \infty\}$) *if and only if* $|\psi\rangle$ *is an eigenstate of A.*

(b) If $\mathcal{A} \sim \{A_i\}$ *is a POVM implemented by the Lüders instrument* $\Phi^{\mathcal{A}}_{\mathcal{L}}$, $\mathfrak{D}_\alpha(\mathcal{A};\,|\psi\rangle) = 0$ ($\alpha \in \{1, F, \infty\}$) *if and only if* $|\psi\rangle$ *is a common eigenstate of the operators* $\{A_i\}$[32,33].

(c) If $\mathcal{A}$ *is a POVM implemented by a general CP instrument* $\Phi^{\mathcal{A}}(\rho) = \sum_i K_i \rho K_i^\dagger$, $\mathfrak{D}_\alpha(\mathcal{A};\,|\psi\rangle) = 0$ ($\alpha \in \{1, F, \infty\}$) *if and only if the state* $|\psi\rangle$ *satisfies*

$$\sum_i |\langle\psi\,|\,K_i\,|\,\psi\rangle|^2 = 1.$$

Disturbance trade-off for projective measurements

From the condition for zero-disturbance for pure states in a projective measurement, we observe the following. For a pair of observables A and B with purely discrete spectra, define the quantity

$$d_\alpha(A,B) \equiv \inf_{|\psi\rangle} \frac{1}{2}[\mathfrak{D}_\alpha(A;|\psi\rangle) + \mathfrak{D}_\alpha(B;|\psi\rangle)].$$

Then, $0 \le d_\alpha(A, B) \le 1$ ($\alpha \in \{1, F, \infty\}$), with $d_\alpha(A, B) = 0$ if and only if A and B have a common eigenvector. Thus we have the following disturbance trade-off principle:

For any two observables A and B with purely discrete spectra which do not have any common eignevector, there exists a quantity $d_\alpha(A, B) > 0$*, such that for any pure state* $|\psi\rangle$*, the average of the disturbances due to measurements of A and B (performed independently, on identically prepared copies of* $|\psi\rangle$*) is greater than or equal to* $d_\alpha(A, B)$*.*

A general disturbance trade-off for a set of observables $\{A_1, A_2, \ldots, A_N\}$ is a *state-independent* lower bound of the form

$$\frac{1}{N}\sum_{i=1}^{N} \mathfrak{D}_\alpha(A_i;|\psi\rangle) \ge d_\alpha(A_1,\ldots,A_N), \quad \forall\,|\psi\rangle,$$

where $d_\alpha(A_1,\ldots, A_N) > 0$ whenever the set of observables $\{A_i\}$ do not have any common eigenvector.

The above disturbance trade-off principle holds *only* for pure state ensembles. If we take into consideration mixed states as well, then there is no non-trivial lower bound. In finite-dimension d, we have the *maximally mixed state* $\mathbb{I}/d$, which is *not disturbed* by the measurement of any observable, irrespective of the disturbance measure used.

Finally, we note that, although we have formulated the trade-off principle using a specific class of distance measures D_α, such a trade-off principle holds for *any* disturbance measure which is based on a distance $D(\rho, \sigma)$ satisfying $D(\rho, \sigma) = 0$ iff $\rho = \sigma$.

Disturbance and uncertainty

The disturbance trade-off principle for projective measurements seems to bear a close resemblance to the well-known uncertainty trade-off principle; the lower bounds in both cases vanish iff the set of observables has a common eigenvector. However, both conceptually and mathematically, the notions of disturbance and uncertainty associated with a measurement are very different. Indeed, the disturbance measures in eq. (6) do not involve the *probabilities* for obtaining different outcomes in a measurement; whereas the entropies which are used to quantify uncertainty, measure the spread in the probability distribution over the outcomes.

Thus, there is no obvious relation between the disturbance caused by a measurement and the uncertainty over its outcomes. However, for projective measurements, the eigenstates of the observable are the states which are left undisturbed by the measurement, and they are *also* the states in which the spread of the probability distribution is zero. Therefore, for projective measurements, the set of pure states with zero disturbance coincides with the set of zero uncertainty states.

Further, it is easy to see that there is a mathematical *equivalence* between the fidelity-based disturbance measure and the Tsallis entropy T_2, for the case of projective measurements.

$$\mathfrak{D}_F(\mathcal{A};|\psi\rangle) = 1 - \sum_i (p^{A}_{|\psi\rangle}(i))^2 = T_2(A;|\psi\rangle). \quad (7)$$

This interesting equivalence between the fidelity-based measure of disturbance and the uncertainty measure given by the T_2 entropy holds *only* for pure states. For mixed states, the disturbance $\mathfrak{D}_F(A;\rho)$ is in general less than $T_2(A;\rho)$.

Using the equivalence in eq. (7), we can directly obtain disturbance trade-off inequalities for those classes of observables for which a T_2 EUR can be obtained.

Disturbance trade-off for mutually unbiased bases

Let $\mathcal{B}_m \equiv \{|i_m\rangle, i = 1, \ldots, d\}$ $(m = 1, \ldots, N)$ denote a set of N mutually unbiased bases (MUBs) in d-dimensions. Recall that two bases $\mathcal{B}_m$, $\mathcal{B}_n$ are said to be *mutually unbiased* if their respective basis vectors satisfy

$$|\langle i_m | j_n\rangle|^2 = \frac{1}{d}, \quad \forall i, j.$$

Any set of N MUBs in d-dimensions satisfies the following disturbance trade-off relation

$$\frac{1}{N}\sum_{m=1}^{N}\mathfrak{D}_F(\mathcal{B}_m;|\psi\rangle) \geq \left(1-\frac{1}{N}\right)\left(1-\frac{1}{d}\right).$$

In dimensions where a *complete* set of $(d+1)$ MUBs exists, the disturbance trade-off relation becomes an exact equality for the complete set of $(d+1)$ MUBs. The above relation can be obtained as a direct consequence of the following bound due to Wu *et al.*[34]

$$\sum_{i=1}^{d}\sum_{m=1}^{N}(p_{|\psi\rangle}^{\mathcal{B}_m}(i))^2 \leq 1+\frac{N-1}{d},$$

where $p_{|\psi\rangle}^{\mathcal{B}_m}(i) = |\langle i_m|\psi\rangle|^2$ denotes the probability of obtaining the ith outcome while measuring $\mathcal{B}_m$ on a pure state $|\psi\rangle$.

Disturbance trade-off for qubit observables

For any pair of observables in a two-dimensional Hilbert space, we have the following *optimal* disturbance trade-off relation[5].

Theorem 5: *For a pair of qubit observables A, B with spectral decompositions $A = \sum_{i=1}^{2} a_i |a_i\rangle\langle a_i|$, $B = \sum_{j=1}^{2} b_j |b_j\rangle\langle b_j|$, and any pure state $|\psi\rangle \in \mathbb{C}^2$,*

$$\frac{1}{2}[\mathfrak{D}_F(A;|\psi\rangle)+\mathfrak{D}_F(B;|\psi\rangle)] \geq \frac{1}{2}(1-c^2), \tag{8}$$

where $c \equiv \max_{i,j=1,2}|\langle a_i|b_j\rangle|$.

The question of finding the lower bound on the average disturbance simplifies considerably once we use the Bloch sphere representation for qubit observables. In other words, we parameterize A and B in terms of unit vectors $\vec{a},\vec{b} \in \mathbb{R}^3$ and real parameters $\{\alpha_i, \beta_i\}$ as follows: $A = \alpha_1\mathbb{I} + \alpha_2\vec{a}\cdot\vec{\sigma}$ and $B = \beta_1\mathbb{I} + \beta_2\vec{b}\cdot\vec{\sigma}$. The quantity c is then given by

$$c = \sqrt{\frac{1+\vec{a}\cdot\vec{b}}{2}}, \quad (\vec{a},\vec{b} > 0);$$

$$c = \sqrt{\frac{1-\vec{a}\cdot\vec{b}}{2}}, \quad (\vec{a},\vec{b} < 0).$$

We refer to Mandayam and Srinivas[5] for the further details of the proof.

We can also show that this bound is *tight*. When $(\vec{a}\cdot\vec{b})^2 = 1$, $c = 0$, A and B commute and the RHS reduces to 0. This lower bound is attained for the common eigenstates of A, B. When $\vec{a}\cdot\vec{b} = 0$, $c = \frac{1}{\sqrt{2}}$ A and B are mutually unbiased. The bound is 1/4, which is attained for any eigenstate of A or B. For any other value of $\vec{a}\cdot\vec{b}$, the lower bound is attained for the states whose Bloch vectors bisect the angle between $\vec{a}$ and $\vec{b}$. The minimizing states are thus given by

$$|\psi_\pm\rangle\langle\psi_\pm| = \frac{1}{2}\left(\mathbb{I}+\left[\frac{\vec{a}\pm\vec{b}}{2c}\right]\cdot\vec{\sigma}\right).$$

Since our disturbance measure $\mathfrak{D}_F(A;|\psi\rangle)$ is in fact the same as the entropy $T_2(A;|\psi\rangle)$, the trade-off relation in eq. (8) is nothing but a tight entropic uncertainty relation for the T_2 entropy:

$$\frac{1}{2}[T_2(A;|\psi\rangle)+T_2(B;|\psi\rangle)] \geq \frac{1}{2}(1-c^2). \tag{9}$$

Our result for T_2 assumes importance in the light of the fact that such optimal analytical bounds are known only for a handful of entropic functions, namely the Rényi entropies H_2 (ref. 35), $H_{1/2}$, and the Tsallis entropy $T_{1/2}$ (ref. 20). For the Shannon entropy, there is in general only a numerical estimate of the bound[15,24].

Disturbance trade-off for non-projective measurements

In this section we consider the more general class of discrete observables characterized by POV measures and associated measurement transformations characterized by CP instruments. While the uncertainty trade-off for a pair of POVM observables depends only on the positive operators characterizing the observables (see Lemma 2 above), the associated disturbance trade-off crucially depends on the CP instruments which implement the measurements of these observables. We state our result concerning the disturbance trade-off principle for such observables here and refer to Mandayam and Srinivas[5] for the proof.

Theorem 6: *For a pair of discrete POVMs $\mathcal{A} \sim \{A_i\}$ and $\mathcal{B} \sim \{B_j\}$, whose measurements are implemented by appropriate CP instruments $\Phi^{\mathcal{A}}$ and $\Phi^{\mathcal{B}}$, the average disturbance*

$$d_\alpha(\mathcal{A},\mathcal{B}) \equiv \inf_{|\psi\rangle} \frac{1}{2}[\mathfrak{D}_\alpha(\mathcal{A};\ |\psi\rangle) + \mathfrak{D}_\alpha(\mathcal{B};\ |\psi\rangle)],$$

satisfies $0 \le d_\alpha(\mathcal{A}, \mathcal{B}) \le 1$ ($\alpha \in \{1, F, \infty\}$). Further, if the measurements of $\mathcal{A}$ and $\mathcal{B}$ are implemented by Lüders channels $\Phi^{\mathcal{A}}_{\mathcal{L}}$ and $\Phi^{\mathcal{B}}_{\mathcal{L}}$, then,

(i) $d_\alpha(\mathcal{A}, \mathcal{B}) = 0$ if and only if all the positive operators $\{A_1, A_2, \ldots, B_1, B_2, \ldots\}$ have a common eigenvector.

(ii) For any suitable entropy measure S, if $c_S(\mathcal{A}, \mathcal{B})$ is the uncertainty trade-off, then, $c_S(\mathcal{A}, \mathcal{B}) = 0 \Rightarrow d_\alpha(\mathcal{A}, \mathcal{B}) = 0$, but not vice versa.

Our result thus implies that the disturbance trade-off for a pair of Lüders POVMs vanishes whenever the corresponding entropic uncertainty trade-off vanishes, but not vice versa. We may illustrate this further by comparing the fidelity-based disturbance measure $\mathfrak{D}_F(\mathcal{A};\ |\psi\rangle)$ for a POVM $\mathcal{A}$ implemented by a Lüders channel $\Phi^{\mathcal{A}}_{\mathcal{L}}$ measured on state $|\psi\rangle$, with the T_2 entropy of the corresponding probability distribution. Unlike in the case of projective measurements, the fidelity-based disturbance is less than or equal to the Tsallis T_2 entropy for a POVM $\mathcal{A}$ measured via a Lüders Channel

$$\mathfrak{D}_F(\mathcal{A};\ |\psi\rangle) \le 1 - \sum_i \langle\psi | A_i | \psi\rangle^2 = T_2(A;\ |\psi\rangle).$$

Furthermore, using measurements which are more general than the Lüders class, we can construct examples where the uncertainty trade-off $c_S(\mathcal{A}, \mathcal{B})$ vanishes for a particular state $|\psi\rangle$, but the disturbances $\mathfrak{D}_\alpha(\mathcal{A}; |\psi\rangle)$ and $\mathfrak{D}_\alpha(\mathcal{B}; |\psi\rangle)$ are both strictly positive.

Example 1: ($d_\alpha(\mathcal{A}, \mathcal{B}) = 0 \not\Rightarrow c_S(\mathcal{A}, \mathcal{B}) = 0$).

Let $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle \in \mathcal{H}_3$ be an orthonormal basis for a three-dimensional Hilbert space $\mathcal{H}_3$. Consider the POVM $\mathcal{A} = \{A_1, A_2, A_3\}$.

$$A_1 = \frac{1}{6}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}(|\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3|),$$

$$A_2 = \frac{2}{3}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}(|\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3|),$$

$$A_3 = \frac{1}{6}|\phi_1\rangle\langle\phi_1|.$$

Consider a different POVM $\mathcal{B}$ constructed with vectors $\{|\phi_1\rangle, |\phi_2'\rangle, |\phi_3'\rangle\}$, which form another orthonormal basis for $\mathcal{H}_3$.

$$B_1 = \frac{1}{6}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}(|\phi_2'\rangle\langle\phi_2'| + |\phi_3'\rangle\langle\phi_3'|),$$

$$B_2 = \frac{2}{3}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}(|\phi_2'\rangle\langle\phi_2'| + |\phi_3'\rangle\langle\phi_3'|),$$

$$B_3 = \frac{1}{6}|\phi_1\rangle\langle\phi_1|.$$

$|\phi_1\rangle$ is a common eigenvector of all the operators $\{A_i, B_j, i, j = 1, 2, 3\}$. Hence, measurements of $\mathcal{A}, \mathcal{B}$ via Lüders instruments lead to a zero disturbance trade-off in state $|\phi_1\rangle$. Therefore, $d_\alpha(\mathcal{A}, \mathcal{B}) = 0$. However, since none of the POV elements $\{A_i, B_j,\ i, j = 1, 2, 3\}$ has eigenvalue 1, there is *no state* with a zero uncertainty trade-off. Therefore, $c_S(\mathcal{A}, \mathcal{B}) \neq 0$. $\mathcal{A}, \mathcal{B}$ is thus an example of a pair of POVMs which has a zero disturbance trade-off, but at the same time a non-zero uncertainty trade-off.

Concluding remarks

The above example clearly shows that $d_\alpha(\mathcal{A}, \mathcal{B}) = 0 \not\Rightarrow c_S(\mathcal{A}, \mathcal{B}) = 0$. Similarly, we have instances of general non-projective measurements where uncertainty trade-off vanishes in a state in which the disturbance trade-off is non-zero. The disturbance trade-off and the uncertainty trade-off are thus two *distinct* principles which reflect different aspects of complementarity between quantum measurements.

Our work thus brings to light an interesting aspect of complementarity in quantum theory, namely that over and in addition to the trade-off in uncertainties, there is a trade-off in the measurement-induced disturbances also. It will be interesting to investigate whether the lower bounds on the sum of disturbances for specific sets of observables assume further operational significance in the context of quantum cryptography.

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