

Modelling in the spirit of Markowitz portfolio theory in a non-Gaussian world

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Most financial markets do not have rates of return that are Gaussian. Therefore, the Markowitz mean variance model produces outcomes that are not optimal. We provide a method of improving upon the Markowitz portfolio using value at risk and median as the decision-making criteria.

Keywords: Financial markets, median, portfolio theory, value at risk.

Introduction: a short history of portfolio theory

PORTFOLIO diversification has been a theme for the ages. In the *Merchant of Venice*, William Shakespeare had Antonio say: ‘My ventures are not in one bottom trusted, Nor to one place; nor is my whole estate Upon the fortune of this present year’ (source: <http://shakespeare.mit.edu/merchant/merchant.1.1.html>, quoted in ref. 1). Shakespeare wrote it during 1596–98. As Markowitz¹ noted, even Shakespeare knew about covariance at an intuitive level.

A similar sentiment was echoed by R. L. Stevenson in *Treasure Island* (1883), where Long John Silver commented on where he keeps his wealth, ‘I puts it all away, some here, some there, and none too much anywheres...’ (source: <http://www.cleavebooks.co.uk/grol/steven/island-11.htm>). This is, of course, a classic example of diversification.

Not all writers had the same belief about diversification. For example, Mark Twain had Pudd’nhead Wilson say: ‘Put all your eggs in the one basket and – watch that basket’ (Twain, M., 1893, chap. 15). Curiously, Twain was writing the novel to sell it to stave off bankruptcy.

Measuring ‘average’ returns to value a lottery has been around for millennia. But it was Bernoulli² who found that decisions based on just the average return of a random variable lead to problems – famously – the St Petersburg paradox, where a decision-maker rejects a gamble with an infinitely large average payoff. Bernoulli² too made a remark about diversification: ‘... it is advisable to divide goods which are exposed to some small

danger into several portions rather than to risk them all together’.

The concept of risk for one asset has been summarized as the standard deviation for several centuries. It has the advantage of being measured in the same units as the original variable. This measure of risk for evaluating single assets in isolation was suggested by Irving Fisher³. He even commented on the time it takes to compute the standard deviation in different ways.

Markowitz⁴, in the preamble of his famous paper noted the following: ‘We next consider the rule that the investor does (or should) consider the expected return a desirable thing and the variance of return an undesirable thing. This rule has many sound points, both as a maxim for, and hypothesis about, investment behavior.’

We can ask the following questions: Why not consider other measures of central tendency or other measures of variability? The justification is easy for using the mean as a measure of central tendency, if the random variable (here, the rate of return of a portfolio of assets) follows a symmetric distribution (with finite central moments). In that case, other measures of central tendency such as the mode and the median are the same as the mean. In addition, in such cases, a variability measure like the semi-variance (variance is calculated using squared deviations from the mean, whereas semi-variance is calculated by taking squared deviation from any arbitrarily chosen value) is easy to calculate if the distribution happens to be normal. Once we permit skewed distribution, alternatives such as the median as a measure of central tendency and general quantile measures or specific measures such as the value at risk (VaR) or tail VaR may be more reasonable measures of risk. These are essentially measures of risk beyond (or below) a certain threshold.

A brief overview of the Markowitz model

Suppose we have n individual stocks in the portfolio. We can work out the return and risk of the portfolio for all combinations of different proportions invested in different individual stocks. It produces an ‘envelope’ that Markowitz called the ‘efficient set’. In Figure 1, the shaded area represents all possible combinations of individual stocks. The parabola represents the envelope.

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Markowitz⁴ recognized that the efficient set need not be a single parabola but a series of interconnected parabolas. In textbooks, they are often represented as one smooth surface without any ‘edges’. Markowitz⁴ erroneously had a non-convex segment. Also, Markowitz⁵ proved a theorem ruling out such non-convex segments. In fact, he¹ declared that figure 6 of his article⁴ was wrong for that reason.

In the original exposition of Markowitz⁴, there were several insights that were new. First, Markowitz explicitly recognized that for a portfolio of assets, a combination of individual stocks will produce a range of feasible asset allocations (shown in the shaded region in Figure 1). The frontier generated by all possible combinations of the assets will be smooth only when n is infinitely large (otherwise, it will be piecewise parabolic). This becomes important when we consider the capital allocation line (CAL, which is also at times referred to as the capital market line, CML), as it may produce a non-unique solution to the ‘market portfolio’. Second, Markowitz explicitly noted that simply applying Bernoulli^{5,6} to claim that the law of large numbers implies that the portfolio risk can be eliminated is wrong: ‘... the law of large numbers applies to a portfolio of securities cannot be accepted. The returns from securities are too inter-correlated. Diversification cannot eliminate all variance.’ Third, Markowitz showed that to an investor, what is most important is not the risk of a given asset measured by its variance, but the contribution the asset makes to the variance of the entire portfolio: It is a question of its covariance with all the other securities in his portfolio. Markowitz constructed what was later called an ‘efficient frontier’ that can be presented to the investor as a menu to choose from.

The mean variance theory proposed by Markowitz requires one of two things: (1) The distribution of the

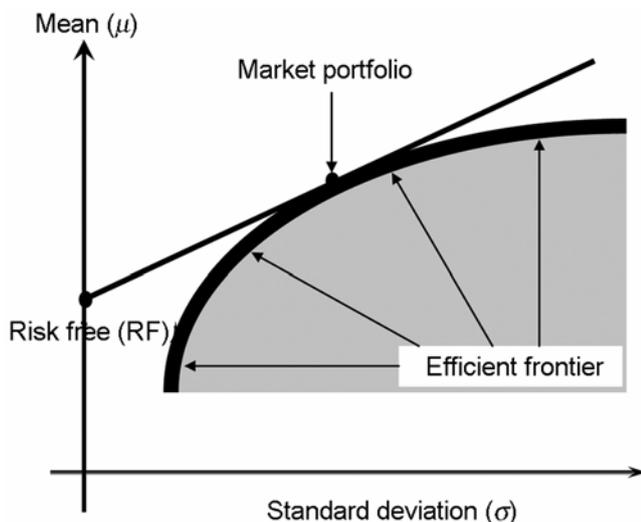


Figure 1. Efficient frontier.

returns should be multivariate normal and the utility function is exponential, or (2) the utility function of the investor is quadratic. It has been well-known that the return structure in most countries most of the time does not have multivariate normal structure. It is also well-known that a quadratic utility function implies that more money produces less utility – a stance that economists do not find realistic. Why then did the Markowitz formulation become so popular? According to Elton and Gruber⁷ the intuitive appeal of the model has made it persist even though the axioms behind the model are not consistent with reality.

The model of downside risk management of Andrew Donald Roy

Roy⁸ published a paper in the same year that Markowitz published his seminal work on portfolio theory. Roy’s starting point was also on the basis of the mean (μ) standard deviation (σ) analysis to make investment decisions. He proposed that the investor chooses a portfolio to maximize $[\mu - d]/\sigma$, where d is some fixed minimum acceptable level of return. Roy called it the ‘safety first principle’ (he considered d to be the disaster level). In order to achieve that, Roy postulated that a sufficient condition is to minimize the probability of going below d . In fact, if the distribution of the portfolio returns is multivariate normal, then the two conditions (of minimizing the probability of disaster and maximizing the ratio $[\mu - d]/\sigma$) are equivalent.

Roy was the first to emphasize the downside risk of a distribution in the context of an investment. The standard deviation (or equivalently, variance) is a global measure of variability. It gets bigger whether the deviation (from the mean) is to the left or to the right. If our penalty function predicates that higher variability is bad, then any upside risk and a downside risk of equal magnitude are equally undesirable. This formulation is clearly unrealistic for investment purposes. After all, why would an investor be equally averse to a gain and to a loss?

It has been speculated that Roy’s model was motivated by his experience in the Second World War. He volunteered for the British Army and fought in the frontline against the Japanese during the Battle of Imphal in India for the British Army, where he saw large war casualties first hand⁹.

Computational aspects of the Markowitz model: Sharpe and beyond

To apply the Markowitz model, one needs to estimate from the data the mean, variance and covariance of every pair of rates of return that one has. Suppose we have observed N stocks over T periods, there are N means to be calculated along with N variances and $N(N - 1)/2$ covari-

ances. In the New York Stock Exchange (NYSE), there are nearly 3,000 stocks listed. In the Bombay Stock Exchange (BSE), there are over 5,000 stocks listed. Suppose we have 1,000 stocks in our portfolio and we want to apply the Markowitz model. It requires that we compute 1,000 means and 1,000 variances and 499,500 covariances. Suppose we have information that needs to be updated everyday for a portfolio, we need over half a million calculations. During the 1950s, such calculations were a tall order.

Suppose now there is a risk-free asset. In practice, the risk-free asset is assumed to be a short-term bond issued by the Government (such as Treasury Bills in the US). Then, the efficient frontier is represented by the line that is a tangent to the efficient frontier which intersects the vertical axis at the level of the risk-free asset (Figure 1).

This assumption produces the following result (known as the two-fund separation theorem): Investors would choose just two funds, one being the riskless asset and the other is a portfolio of individual stocks – called the market portfolio. Depending on their risk appetite, they will simply choose different proportions of these two assets. The higher the risk aversion, the larger will be the proportion of the riskless asset chosen by the investor. This straight line, on which all investors would choose, has become known as the CML.

It also reduces the computational burden mentioned earlier about calculating a large number of covariances. Once the market portfolio is known, the investor simply needs to calculate how much to invest in just two funds. Of course, in order to calculate the efficient frontier, we do need to calculate all the covariances if new information about the returns comes in.

The introduction of the riskless asset was first proposed by Tobin¹⁰. He only considered the riskless asset of cash which generates no interest at all. The modern formulation become popular with Treasury Bills as riskless assets after Sharpe¹¹ introduced them. We define the Sharpe ratio as follows: for a portfolio P (of n assets) let $\mu(P)$ be the mean return on portfolio P and $\sigma(P)$ be the standard deviation of the portfolio P . Let r denote the return on the riskless asset. Then the Sharpe ratio $s(P)$ for a portfolio P is defined as

$$s(P) = (\mu(P) - r)/\sigma(P).$$

The market portfolio is the portfolio for which the Sharpe ratio is maximized. Under some regularity conditions, the efficient frontier will be strictly convex (from above). In this case, the market portfolio can be guaranteed to be unique.

The notion of the Sharpe ratio was clearly anticipated by Roy⁸, which we discussed earlier. Instead of a riskless asset, Roy⁸ posited a disaster level d , and suggested maximization of $[\mu - d]/\sigma$.

Normal (or Gaussian) distribution and risk management

If the joint distribution of all the assets has a multivariate normal distribution, then the distribution of any portfolio constructed out of a linear combination of those assets also has a normal distribution. Therefore, the risk can be measured by its variance (or equivalently by its standard distribution). Suppose that two assets have the same mean return and that the first has a smaller variance than the second, then it can be shown that for any reasonable definition of risk, the first asset will have lower risk than the second provided both assets have normal distributions. However, when the distributions are not Gaussian, the same statement is no longer true.

It has long been recognized that data-generating processes in the markets do not seem to follow normal or Gaussian distributions. Mandelbrot¹² showed that the Gaussian distribution may reasonably capture the shape of the centre of the distribution, but it does not fit the tail of the distribution. Events that are 3–4 standard deviations away from the mean are extremely uncommon in the Gaussian model. For example, an event over three standard deviations away from the mean has a probability of 0.0027. An event of four standard deviations away has a probability of 0.000063. Thus, assuming a Gaussian distribution for the return of a portfolio, an investor might conclude that the price of his assets will not drop by more than the mean minus three standard deviations once in 300 years. Therefore, it gives a false sense of security to the investor, underestimating his or her risk. Indeed, in the risk analysis/risk management literature and in the regulatory framework, it has been generally accepted that returns on stocks often may not follow a Gaussian distribution. Alternate methods of measuring and controlling risk have been developed. The Basel II Agreement prescribes VaR as *the* measure of risk.

Value at risk

The VaR for an asset is defined as follows:

Let L be the loss (for an asset) over a given time period. Then the 1% VaR for the asset is a value v such that the probability that the loss L exceeds v is at most 1%, i.e.

$$P(L \geq v) = 0.01.$$

Thus the 1% VaR v is the upper 1th percentile point (or the 99th percentile) of the loss distribution.

If we accept that VaR as a measure for risk has been accepted by practitioners as well as regulatory authorities, it is natural to use VaR as a measure for risk when dealing with the issue of choosing an optimal portfolio using the risk–reward criterion.

VaR is thus defined as a measure of the potential loss in value of a risky portfolio over a defined period for a given level of confidence. The term was not used widely until the middle of the 1990s, but the concept has been used for a long time. Holton¹³ traces the history of VaR going back to 1922. The Securities and Exchange Commission (SEC) required banks to use VaR in the 1980s; it was called ‘haircuts’. Kenneth Garbade, then at Bankers Trust, used it for traders inside the company starting in 1986. It provided a one number summary of a trader’s position at the end of the day. This was the first use in the current sense of the concept. However, the terminology used was far from standard in the early 1990s. It has been described as ‘dollar at risk’ or ‘capital at risk’ by different companies.

What gave the VaR a widespread boost in the industry was when the J. P. Morgan made public its methodology of calculation of the VaR in 1994. It quickly became an industry ‘gold standard’¹⁴. An important multilateral agreement originating in the European Union (called Basel II) enshrined VaR in 2004 by making it a requirement for the large financial institutions in Europe. The Basel II Agreement prescribes VaR as the measure of risk.

When returns are not Gaussian

When the distributions of the underlying assets are not normal, the decision-making criterion using the mean and variance does not yield the same result as using the mean and the VaR. The reason is that a lower variance or standard deviation does not automatically ensure a lower VaR. We illustrate this phenomenon with an example.

Let the rate of return from a portfolio A of assets be represented by a random variable X with a double exponential (also called Laplace) distribution with a mean of 0.3 and a variance of 1. Let the rate of return from a portfolio B of assets be represented by a random variable Y with a double exponential distribution with a mean of 0.3 and a variance of 1.1 (slightly higher than X). Thus by the mean–variance criterion, B is more risky than A as the rate of return from B has the same mean as the rate of return from A , but a 10% higher variance. However, it can be shown that

$$P(X \leq -2.466) = 0.01 \\ \text{[or equivalently } P(-X \geq 2.466) = 0.01],$$

while

$$P(Y \leq -2.357) = 0.01 \\ \text{[or equivalently } P(-Y \geq 2.357) = 0.01].$$

Thus the 1% VaR for A (99th percentile for $-X$) equals 2.466, while a 1% VaR for B (99th percentile for $-Y$)

equals 2.357. Therefore, A is more risky if we use a 1% VaR as the risk measure. Thus, from a regulatory risk management perspective, B is preferable to A . However, if we use the standard Markowitz mean–variance framework, we would conclude that A is preferable to B .

This serves as the motivation for using the same risk measure in risk management as well as a risk–return optimality criterion in portfolio theory. As we noted above, the VaR is the gold standard for the industry and the regulators for measuring risks.

The measure of VaR has been criticized on technical grounds. For example, Artzner *et al.*¹⁵ argued that a far better measure related to the VaR would be a conditional VaR (CVaR, also called average VaR, tail VaR or expected Shortfall).

CVaR is the expected loss given that a loss event has happened. So the 1% CVaR c is given by

$$c = E(L | L > v),$$

where v is the 1% VaR.

In our example of the double exponential distribution above, the CVaR for the portfolio A is 3.19, while for the portfolio B it is 2.93. Therefore, with CVaR as the risk measure, we should prefer B over A .

Proposed measure of risk and return: VaR and median for non-Gaussian returns

For choosing an optimal portfolio, we propose the use of one measure of risk (either a 1% VaR or 1% CVaR). At the same time, once we move away from Gaussian distributions, we could use the median as the proxy for the ‘average’ return rather than the mean. This will be desirable as it would allow us to consider assets whose distributions may not be symmetric and may not even admit a mean (such as Cauchy distribution)¹⁵. In this article we will use 1% VaR as a measure for risk as this is the common practice in risk management.

Thus we reconsider the Markowitz framework with the median as the proxy for the return on investment and a 1% VaR as the proxy for risk. In principle, we could proceed exactly as in the classical Markowitz framework. Consider all possible portfolios. For each portfolio, compute the median and the VaR. Then choose the portfolio P for which return risk trade-off ratio (or RT ratio for short) defined as:

$$RT(P) = (\text{median}(P) - R)/(\text{VaR}(P)),$$

is maximum, where $\text{median}(P)$ is the median for the portfolio P , $\text{VaR}(P)$ is the 1% VaR for the portfolio P and R is the riskless return (government bonds). Then $RT(P)$ is the analogue of the Sharpe ratio in this new risk–return framework.

If P^* is the portfolio such that $RT(P) \leq RT(P^*)$ for all P , then we can call P^* the optimal portfolio and then the line that joins the riskless asset with the portfolio P^* (or to be precise $(0, R)$ is joined to $(\text{VaR}(P), \text{median}(P))$) is the analogue of CML. Exactly as in the Markowitz mean–variance framework, one can argue that any investor who accepts the median as a proxy for return and the VaR as the proxy for risk should use a convex combination of P^* and the risk-less security as his/her portfolio. In general, the median of a convex combination of two random variables does not equal the corresponding convex combination of medians of the two random variables. However, when one of the two random variables is degenerate, the median of a convex combination of two random variables does equal the corresponding convex combination of medians of the two random variables. Hence we can have the same interpretation to the line joining the risk-free asset and the optimal portfolio P^* as in the mean–standard deviation case. For a given level of median return, the point on the (proposed) CML would minimize the VaR, while for a given level of VaR the point on the line would maximize the median return.

How does one determine the optimal portfolio P^ ?*

First let us observe that if the underlying joint distribution of the returns on the stocks under consideration is a (multivariate) Gaussian distribution, then P^* is the same as the market portfolio in the classical Markowitz mean–variance framework. Because then the distribution of returns on any portfolio is Gaussian and hence the median equals the mean and for a fixed mean (μ), the standard deviation (σ) determines the VaR (indeed, the 1% VaR is $2.38\sigma - \mu$).

What if the joint distribution is *not* Gaussian? In the context of returns on stocks, a more realistic model can be constructed using a copula. This allows using distributions with fatter tails (fatter than Gaussian), such as Laplace (double exponential), logistic, Cauchy as the model for the marginal distributions, and the dependence among the returns on the stocks being taken care by the copula. Here too, the Gaussian copula is not appropriate as it underestimates the tail dependence. One should use a t -copula with one or two degrees of freedom. Of course, using copula-based models means that all subsequent estimations (of the median or VaR of a portfolio) would be done using Monte Carlo simulations.

Given historical data, one could estimate the rank correlation among the stock returns and use a t -copula with the estimated rank correlation matrix as the copula. The marginal distributions can be fitted from among a family of distributions that include the well-known distributions, or the empirical marginal distribution can be used.

Once the marginal distributions and copula are chosen, we can simulate observations from the chosen joint dis-

tribution using Monte Carlo techniques. If one is going to use a 1% VaR as the risk measure, the simulation step should be used to generate at least 10,000 observations from the joint distribution, as a smaller number of observations may not ensure the reliability of a 1% tail.

Since we do not have an algebraic way of dealing with the question of obtaining the market portfolio P^* , we could consider all possible portfolios p_1, p_2, \dots, p_n , where each p_j is a multiple of a fixed number $\delta = 1/M$, where n is the number of stocks under consideration. For a given portfolio, one can determine the median as well as the VaR by first computing the returns for the portfolio (in the simulated data) and then sorting them.

This modest proposal can bring an exploding computational difficulty. For example, when $n = 30$ and we only take $\delta = 0.01$, the total number of portfolios exceeds a trillion trillion and so an exhaustive search is ruled out. Indeed, the total number of portfolios with only integral percentage components exceeds 2^{95} .

Even if one had a billion computers each working at 1,000 gigahertz and an efficient code that computes $RT(P)$ in a single clock cycle, it will take over 1,900 years to compute $RT(P)$ for all possible portfolios (where each p_j is a multiple of 0.01). So we propose an alternate method for doing such calculations.

Proposal for computing near optimal portfolios

We propose that one could start with the Markowitz market portfolio $P = (p_1, p_2, \dots, p_n)$. We could consider random perturbations of P to generate a large number (say 1,000) of portfolios and pick the top 50 portfolios among the 1,001 portfolios (the top 50 according to the criterion of giving largest $RT(P)$). Once again, we consider random permutations, say 100 permutations of each of the 50 portfolios to generate a total of $50 + 5,000$ portfolios and again pick the top 50 portfolios. We can repeat this step, say six times and then pick the portfolio that yields the largest $RT(P)$. We may not have gotten the optimal solution, but perhaps this will generate a near-optimal solution. In any case, by this method, we generate a portfolio that is better than the Markowitz market portfolio.

We could even start with a basket containing the Markowitz market portfolio and a few other portfolios that have been generated via other methods. We outline the proposed method below.

Proposed algorithm and the pseudo-code

Let n be the number of stocks under consideration. We will represent portfolios by P, Q, R, \dots each of which will be an n -dimensional vector whose components are non-negative and the sum of the components is 1.

We will consider the following perturbations of the portfolios. Perturbations will be of two kinds: additive

and multiplicative. For portfolios P, Q and a number α between 0 and 1, let $A(P, Q, \alpha) = (1 - \alpha)P + \alpha Q$ denote the additive perturbation of P . The multiplicative perturbation $M(P, Q, \alpha) = R$ is defined as follows: Let $P = (p_1, p_2, \dots, p_n)$, $Q = (q_1, q_2, \dots, q_n)$. Let $r_j^* = (1 - \alpha + \alpha q_j) p_j$ and $r_j = r_j^* / (r_1^* + r_2^* + \dots + r_n^*)$. Then $M(P, Q, \alpha) = R = (r_1, r_2, \dots, r_n)$.

The difference between additive and multiplicative perturbations is that while the first perturbs all components of the vector, the latter only perturbs the non-zero components.

We will choose integers k, m, s, t and a number α between 0 and 1. k is the initial number of perturbations of the market portfolio taken in the initial step, m the number of portfolios kept across the iteration, s the number of perturbations of the portfolios in each iteration step, t the number of iterations and α is the perturbation parameter (say $k = 1000, m = 50, s = 50, t = 7$ and $\alpha = 0.2$).

In order to generate a random portfolio R , we proceed as follows. Let U_1, U_2, \dots, U_n be independent and identically distributed uniform $[0, 1]$ random variables and let $R_j = U_j / (U_1 + \dots + U_n)$. Then $R = (R_1, \dots, R_n)$ would represent a random portfolio. Each time we need to generate a random permutation, we generate it independently of previous choices.

Step 1: Let P^0 be the Markowitz market portfolio. For $i = 1, 2, 3, \dots, k$, let R^i be a random portfolio, generated as

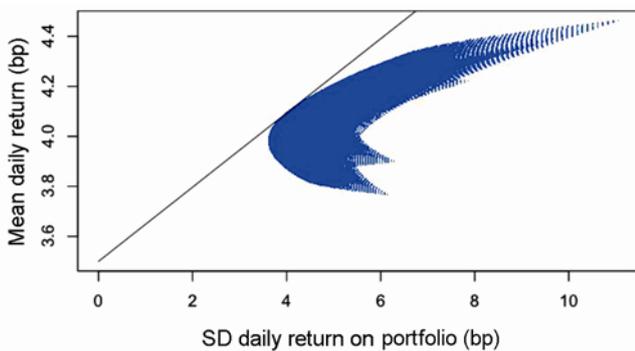


Figure 2. Mean standard deviation efficient frontier with simulated data.

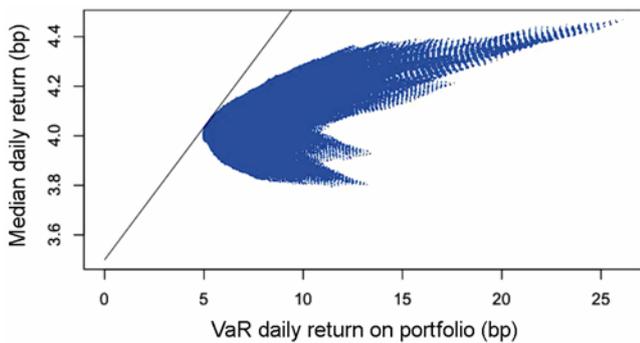


Figure 3. Median value at risk efficient frontier with simulated data.

described above (generated independently for each i) and let $P^i = M(P^0, R^i, \alpha)$ for $1 \leq i \leq k/2$ and $P^i = A(P^0, R^i, \alpha)$ for $k/2 < i \leq k$. For each P^i , let us compute $RT(P^i)$ and then order them in decreasing order by $RT(P^i)$. Let $P^{*1}, P^{*2}, \dots, P^{*k}$ be a reordering of P^1, P^2, \dots, P^k such that $RT(P^{*1}) \geq RT(P^{*2}) \geq \dots \geq RT(P^{*k})$, and let $Q^0 = P^0$ and $Q^i = P^{*i}$ for $i = 1, 2, 3, \dots, m$. Thus, Q^1, Q^2, \dots, Q^m are the m -best portfolios among P^1, P^2, \dots, P^k here best is in the sense of higher $RT(P)$. Set $a = 1$ (a is the number of times the iteration has been performed).

Step 2: For $0 \leq i \leq m$, let $P^{i,0} = Q^i, P^{i,j} = M(Q^i, R^{i,j}, \alpha), 1 \leq j \leq s/2$ and $P^{i,j} = A(Q^i, R^{i,j}, \alpha), s/2 < j \leq s$, where $R^{i,j}$ are random portfolios (generated independently of previous choices).

Once again we take the top m portfolios amongst $\{Q^{i,j}: 0 \leq i \leq m, 0 \leq j \leq s\}$ that yield the largest $RT(P)$. Let us call these k portfolios Q^1, Q^2, \dots, Q^m and $Q^0 = P^0$.

Step 3: Let $a = a + 1$ and replace α by $\alpha/2$. If $a > t$ go to step 4 else go to step 2.

Step 4: Let P^* be the portfolio among $Q^0, Q^1, Q^2, \dots, Q^m$ that maximizes $RT(Q)$. P^* is then the market portfolio.

A numerical example

We consider four stocks. The returns on the four stocks (S^1, S^2, S^3, S^4) are modelled as having a joint distribution characterized as follows: the marginal distributions of each of S^j are double exponential (or Laplace distribution) with means 3.8, 3.9, 4.2, 4.5 and standard deviations 6.1, 6.3, 7.9, 11.1 respectively (here the unit is the basis point). The joint distribution is then determined by a t -copula with one degree of freedom and the rank correlation matrix given as follows.

	S^1	S^2	S^3	S^4
S^1	1			
S^2	0.66	1		
S^3	0.2	0.33	1	
S^4	0.23	0.3	0.74	1

Suppose the riskless return is 3.5 (basis points). For this distribution, the portfolio that maximizes the Sharpe ratio is given by $P^\# = (0.192962, 0.272929, 0.318704, 0.215406)$. For this portfolio, $\mu(P^\#) = 4.09801, \sigma(P^\#) = 4.02967$ and $s(P^\#) = 0.148401$. Median ($P^\#$) = 4.10525 and the 1% VaR ($P^\#$) = 5.97985. The tau ratio $RT(P^\#) = 0.101215$ (Figures 2 and 3).

For the same distribution, the market portfolio in the proposed median-VaR framework is given by $P^* = (0.238056, 0.327227, 0.265115, 0.169602)$. For this portfolio $\mu(P^*) = 4.04902, \sigma(P^*) = 3.76794$ and $s(P^*) = 0.145708, \text{median}(P^*) = 4.06861, \text{VaR}(P^*) = 5.29185$ and $RT(P^*) = 0.107449$.

Therefore, our method produces a 6% improvement of the tau ratio (the Sharpe ratio equivalent).

Conclusion

It has been demonstrated time and again that in many financial markets, the rates of return of many assets are far from Gaussian. In particular, in commodity markets, the markets for exchange rates have large and persistent deviations from normal. In addition, even the rates of return in stock markets with thin trading have fatter tails than normal. This makes variance as a risk measure misleading. On the other hand, value at risk rather than variance has become the standard measure of risk both in the market and for the regulators. We offer a practical solution to deal with both of these problems together.

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