# Credit portfolio optimization with replacement in defaultable asset

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**In this article, we propose a new model for credit portfolios. We prove the existence of optimal portfolios for the associated power utility maximization problem and the benchmarked optimization problem.**

**Keywords:** Bench-marked optimization, credit portfolio, defaultable asset, power utility.

## Introduction

In this article, we introduce a new portfolio model of money market account, a stock and a defaultable asset. Credit portfolio models consisting of money market account, stock and defaultable asset have already been studied by various authors[1,2]. But all these models assume that upon default (i.e. of the defaultable asset), the portfolio continues with the money market account and the stock till maturity. One existing method to circumvent this situation is to incorporate large number of defaultable assets in the portfolio and look at the large portfolio asymptotics; see Sircar and Zariphopoulou[3] and the references therein. This approach in general leads to an infinite dimensional value process for the portfolio wealth. We propose a different approach. Our method is to replace the defaultable asset with another defaultable asset upon each time of default. By doing this, we arrive at a portfolio optimization model, where the dynamics of the value process is given by a Markov switching jump diffusion model (see eq. (3) later), which is finite dimensional. There exists literature on portfolio optimization using jump diffusion models[4,5]. None of these models incorporates switching in the diffusion.

In general there are three approaches to portfolio management, utility maximization of portfolio wealth; benchmark optimization and mean–variance optimization. Using our portfolio model, we study utility maximization using power utility and benchmark optimization problem.

This article is structured as follows. First, we describe the portfolio model and the portfolio optimization problems. The utility maximization problem is addressed next. Then, we prove the existence of the optimal portfolio. In the next section, we prove the existence of optimal benchmarked portfolio, followed by some numerical results.

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## Portfolio model and problem description

We consider a financial market with a basket of defaultable assets, one stock and a risk-free security called money market account. At any given time the basket of defaultable assets contains $M$-type of assets with $i$th type being identified by a pair of parameters $(b(i), \bar{\sigma}(i))$, where $b(i)$ denotes the expected rate of return and $\bar{\sigma}(i)$ denotes the volatility vector, $i = 1, 2, \ldots, M$. The asset price evolution until default of any $i$th type defaultable asset from the basket is given by the stochastic differential equation (SDE in short),

$$\mathrm{d}S_t(t) = S_i(t)\left[b(i)\mathrm{d}t + \sum_{j=1}^{n}\sigma(i, j)\mathrm{d}W_j(t)\right], \; i = 1, 2, \ldots, M,$$

where $\bar{\sigma}(i) = (\sigma(i, 1), \sigma(i, 2), \ldots, \sigma(i, n))$ and $W(\cdot) = (W_1(\cdot), W_2(\cdot), \ldots, W_n(\cdot))$ is a standard $\mathbb{R}^n$-valued Wiener process on the complete probability space $(\Omega, F, P)$.

The evolution of the stock price is given by

$$\mathrm{d}S(t) = S(t)\left[\hat{b}\,\mathrm{d}t + \sum_{j=1}^{n}\hat{\sigma}(j)\,\mathrm{d}W_j(t)\right].$$

The money market account has a price $S_0(\cdot)$ which evolves according to the equation,

$$\mathrm{d}S_0(t) = S_0(t)r\,\mathrm{d}t,$$

where $r$ is the interest rate of the money market account.

Our portfolio model at any given time consists of a single defaultable asset chosen from the basket, the stock and the money market account. The evolution of the portfolio over time is as follows. Upon default of the defaultable asset in the portfolio, investors choose another defaultable asset from the basket as a replacement and the procedure continues up to the terminal time $T$. The default time and the replacement of defaulted asset in the portfolio is according to a continuous time Markov chain $Y(\cdot)$ with state space $S = \{1, 2, 3, \ldots, M\}$ and rate matrix $\Pi = (\pi_{ij})_{M\times M}$, i.e. the default time is modelled by the switch time of $Y(\cdot)$, and when the Markov chain is in the state $i$ the portfolio contains the $i$th type defaultable asset.

Then the evolution of $Y(\cdot)$ is given by

$$P(Y)(t+\delta t) = j|Y(t) = i) = \begin{cases} \pi_{ij}\delta t + o(\delta t) & \text{if } i \neq j \\ 1+\pi_{ii}\delta t + o(\delta t) & \text{if } i = j, \end{cases} \quad (1)$$

where

$$\pi_{ij} := \begin{cases} \lambda_i p_{ij} & \text{if } i \neq j \\ -\lambda_i & \text{otherwise.} \end{cases} \quad (2)$$

**Remark 1.** *The basket of assets contains M-type of defaultable assets with multiple number of each type of asset in the basket. Hence even if the ith type defaultable asset is defaulted at a time, the basket of assets will contain the ith type defaultable asset. The model as it stands now will not prevent keeping an asset in the portfolio even if the asset of the same type was defaulted in the past. The model only prevents the same type of asset in the portfolio for immediate replacement at default. Rectifying this issue is an interesting future work.*

We assume that Markov chain $Y(\cdot)$ and the Wiener process $W(\cdot)$ are independent. Let $\mathcal{F}_t$ be the right continuous augmentation of $\sigma(W(s), Y(s), 0 \leq s \leq t)$.

Let $\tau_n$ denote the time of the $n$th switch of the Markov chain $Y(\cdot)$, i.e. the time of the $n$th default in the portfolio is $\tau_n$, $n \geq 1$. Set

$$I_n = \tau_n - \tau_{n-1},\ n \geq 1,\ I_0 = 0,\ \tau_0 = 0,$$

and

$$N(t) = \inf\{n \geq 0 \mid I_n \leq t, I_{n+1} > t\}.$$

Then $N(\cdot)$ is a Cox process with intensity $\lambda(Y(t))$. For more details, one can refer to Last and Brandt[6]

Consider the process $\bar{S}(\cdot)$ given by

$$\bar{S}(t) = \begin{cases} S_{Y(t)}(t) & \text{if } \ t \neq \tau_i \text{ for any } i, \\ 0 & \text{for } t = \tau_i \text{ for some } i. \end{cases}$$

for all $i \geq 1$.

The process $\bar{S}(\cdot)$ denotes the combined dynamics of the price process of defaultable assets used in the portfolio. Note that $\bar{S}(\cdot)$ is neither a RCLL (right continuous with left limits) nor a LCRL (left continuous with right limits) process. Also note that the right continuous version of the process $\bar{S}(\cdot)$ is $S_Y(\cdot)$.

Consider a portfolio process $\phi(\cdot) = (\phi^0(\cdot), \phi^1(\cdot), \phi^2(\cdot))$, which is predictable and self-financing. Here $\phi^0(t)$, $\phi^1(t)$ and $\phi^2(t)$ denote the number of shares of the money market, stock and the defaultable asset at time $t$ respectively.

The value at time $t$ of the portfolio $\phi(\cdot)$ is given by

$$X(t) = \phi^0(t)S_0(t) + \phi^1(t)S(t) + \phi^2(t)\bar{S}(t),$$

i.e.

$$X(t) = \begin{cases} \phi^0(t)S_0(t) + \phi^1(t)S(t) + \phi^2(t)\bar{S}(t) & \text{if } \ t \neq \tau_i \text{ for any } i, \\ \phi^0(t)S_0(t) + \phi^1(t)S(t) & \text{for } t = \tau_i \text{ for some } i. \end{cases}$$

Under the self-financing condition, the wealth process $X(\cdot)$ is given by the solution of the following switching jump diffusion

$$\frac{dX(t)}{X(t-)} = [r + u_1(t)(\hat{b} - r)dt + u_1(t)(b(Y(t)) - r)]dt$$
$$+\sum_{j=1}^{n}(u_1(t)\hat{\sigma}(j) + u_2(t)\sigma(Y(t), j))dW_j(t)$$
$$-u_2(t)dN(t), \quad (3)$$

where $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ denotes the predictable process representing fraction of wealth invested in the stock and the defaultable asset at time $t$ respectively.

We assume that short selling of the defaultable assets and the stock is not allowed. Hence we restrict our portfolios to the following.

A portfolio process $u(\cdot)$ is said to be admissible and write $u(\cdot) \in \mathcal{A}$ if:

(i) The process $u(\cdot)$ is $A = [0, 1] \times [0, 1 - \varepsilon]$-valued and is predictable with respect to $\{\mathcal{F}_t\}$, where $\varepsilon > 0$ is a fixed constant.

(ii) Equation (3) has a unique weak solution corresponding to the process $u(\cdot)$.

Note that the admissibility condition demands the existence of a unique, weak solution to eq. (3). Condition (ii) is not common when the control does not appear in the driving noise terms. But when the driving noises are controlled, it is generally imposed[7,8]. Further note that, when the control is a prescribed control, then from Protter[9] (theorem 37, p. 84), it follows that eq. (3) has a unique strong solution. Therefore, the set $\mathcal{A}$ of all admissible controls is non empty, because, all prescribed controls are admissible. Also, it is interesting to note that we in fact prove the existence of optimal control which is prescribed; see theorems 2 and 3 below.

Now we will show that the wealth process $X(\cdot)$ corresponding to $u(\cdot) \in \mathcal{A}$ is positive. For an RCLL process $Z(\cdot)$, let $\Delta Z(s)$ denote the jump $Z(s) - Z(s-)$ of the process $Z(\cdot)$ at time $s$. We state the following theorem from Protter[9].

**Theorem 1.** *Let $Z(\cdot)$ be a semimartingale, $Z(0) = 0$. Then there exists a unique semimartingale $X(\cdot)$ that satisfies the equation $X(t) = 1 + \int_0^t X(s-)dZ(s)$. Moreover $X(\cdot)$ is given by*

$$X(t)=\exp\left\{Z(t)-\frac{1}{2}[Z,Z](t)\right\}\prod_{s\le t}(1+\Delta Z(s))$$
$$\times\exp\left\{-\Delta Z(t)-\frac{1}{2}(\Delta Z(t))^2\right\},$$

*where the infinite product converges*.

Let $u(\cdot)$ be an admissible control and $X(\cdot)$ be the corresponding wealth process. Then, from Theorem 1, we have

$$X(t)=\exp\left\{Z(t)-\frac{1}{2}[Z,Z](t)\right\}\prod_{s\le t}(1+\Delta Z(s))$$
$$\times\exp\left\{-\Delta Z(t)-\frac{1}{2}(\Delta Z(t))^2\right\},\ t>0,$$

where

$$\mathrm{d}Z(t)=(r+u_1(t)(\hat{b}-r)\mathrm{d}t+u_2(t)(b(Y(t))-r))\mathrm{d}t$$
$$+\sum_{j=1}^{n}(u_1(t)\hat{\sigma}(j)+u_2(t)\sigma(Y(t),\ j))\mathrm{d}W_j(t)-u_2(t)\mathrm{d}N(t).$$

From the above equation it is clear that $\Delta Z(s) = -u_2(s) > -1+\varepsilon$ at the time of jump; otherwise $\Delta Z(s) = 0$. Therefore, our wealth process is strictly positive for any admissible control.

**Remark 2.** *Portfolio $u(\cdot)$ taking values in A implies no short selling of the stock and the defaultable asset. Also $u_1(\cdot)$, $u_2(\cdot)\le 1$ puts a restriction on borrowing from the money market account. Moreover, further restricting $u_2(\cdot)$ to $[0, 1-\varepsilon]$, prevents putting all money in the defaultable asset.*

We study two types of portfolio optimization problems using our portfolio model described above. In the first problem we look at portfolio optimization using power utility and in the second, we study the benchmark optimization problem. We assume that the time horizon is $[0, T]$.

## *Power utility optimization*

In this portfolio optimization, the objective of the investor is to maximize the terminal expected utility:

$$J_{\theta,T}(x,i,u(\cdot)):=-\frac{1}{\theta}E[(X(T))^{-\theta}|X(0)=x,Y(0)=i],\quad (4)$$

where $X(\cdot)$ is given by eq. (3) corresponding to $u(\cdot)\in\mathcal{A}$.

An investment strategy $u^*(\cdot)$ is said to be optimal if

$$J_{\theta,T}(x,i,u(\cdot))\le J_{\theta,T}(x,i,u^*(\cdot)),\ u(\cdot)\in\mathcal{A}.$$

Note that in general $u^*(\cdot)$ may depend on the initial wealth $x > 0$. The function $f$ defined by

$$f(x,i):=\sup_{u(\cdot)\in\mathcal{A}}J_{\theta,T}(x,i,u(\cdot)),\qquad (5)$$

is called the value function of the optimization problem. Our method to prove the existence of an optimal investment strategy relies on characterizing the value function as a unique solution to certain Hamilton–Jacobi–Bellman (HJB) equations. i.e. we use dynamic programming approach.

## *Benchmarked optimization*

In this portfolio optimization, the objective of the investor is to select a portfolio which outperforms a benchmarked portfolio. Let $L(\cdot)$ denote the price evolution of the benchmark portfolio and is given by the solution of the SDE

$$\mathrm{d}L(t)=L(t)\left[\alpha\,\mathrm{d}t+\sum_{j=1}^{n}\gamma_j\,\mathrm{d}W_j(t)\right],\ L(0)=l,\qquad (6)$$

where $\alpha$ and $\bar{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ are constant.

The investor's objective is to maximize the risk-adjusted growth of his portfolio relative to the benchmark. This objective can be modelled through defining a new optimization criterion, representing the logarithm of the excess return of the asset portfolio over its benchmark, $F(t)$ given by

$$F(t)=\ln\frac{X(t)}{L(t)}=\ln X(t)-\ln L(t).$$

See, for example, Davis and Lleo[10]. By Ito's formula, it follows that $F(\cdot)$ is given by the solution of the SDE

$$\mathrm{d}F(t)=\Bigg[(r+u_1(t)(\hat{b}-r)\mathrm{d}t+u_2(t)(b(Y(t))-r))$$
$$-\left(\alpha-\frac{1}{2}\|\bar{\gamma}\|^2\right)-\frac{1}{2}(u_1^2(t)\|\hat{\sigma}\|^2+u_2^2(t)\|\bar{\sigma}(Y(t-))\|^2$$
$$+2u_1(t)u_2(t)\hat{\sigma}\cdot\bar{\sigma}(Y(t-)))\Bigg]\mathrm{d}t$$
$$+\sum_{j=1}^{n}(u_1(t)\sigma(j)+u_2(t)\sigma(Y(t),\ j)-\gamma_j)\mathrm{d}W_j(t)$$
$$+\ln(1-u_2(t))\mathrm{d}N(t),\qquad (7)$$

where $\hat{\sigma}=(\hat{\sigma}(1),\hat{\sigma}(2),\ldots,\hat{\sigma}(n))$.

The investor's objective is to maximize the terminal expected utility

$$\mathcal{J}_{\theta,T}(z,i,u(\cdot)):=E[-\exp(-\theta(F(T)))\,|\,F(0)=z,Y(0)=i],\qquad (8)$$

over all admissible investment strategies $u(\cdot)$ subject to eq. (7). The definition of value function is as in eq. (5). A portfolio $u^*(\cdot)$ is said to be benchmarked optimal if

$$\mathcal{J}_{\theta,T}(z,i,u(\cdot))\le\mathcal{J}_{\theta,T}(z,i,u^*(\cdot)),\ u(\cdot)\in\mathcal{A}.$$

## Utility maximization

In this section we study the portfolio optimization with constant risk aversion parameter $\theta > 0$, described earlier. For this it suffices to consider the following payoff criterion given by

$$E[(X(T))^{-\theta} \mid X(0) = x,\ Y(0) = i], \tag{9}$$

where $X(\cdot)$ is the solution of eq. (3), since the maximizer of $J_{\theta,T}\ (x,\ i,\ u(\cdot))$ is the minimizer of above criterion and vice versa. Let

$$I_{\theta,T}(t, x, i, u(\cdot)) := E[(X(T))^{-\theta} \mid X(t) = x,\ Y(t) = i],$$

and

$$f(t, x, i) := \inf_{u(\cdot)\in\mathcal{A}} I_{\theta,T}(t, x, i, u(\cdot)), \tag{10}$$

be the corresponding value function. The HJB equation corresponding to the value function $f$ is given by

$$\frac{\partial f}{\partial t}(t, x, i) + \inf_{u\in A}\Bigg[ x(r + u_1(\hat{b} - r) + u_2(b(i) - r)) \times \frac{\partial f}{\partial x}(t, x, i) + \frac{1}{2}x^2(u_1^2 \|\hat{\sigma}\|^2 + u_2^2 \|\bar{\sigma}(i)\|^2 + 2u_1u_2\hat{\sigma}\cdot\bar{\sigma}(i))\frac{\partial^2 f}{\partial x^2}(t, x, i) + \sum_{j\in S}\pi_{ij} f(t, (1-u_2)x, j) + \lambda_i\{f(t, (1-u_2)x, i) - f(t, x, i)\}\Bigg] = 0, \tag{11}$$

with terminal condition

$$f(T, x, i) = x^{-\theta}. \tag{12}$$

We look for a solution of eqs (11) and (12) of the form

$$f(t, x, i) = x^{-\theta} g(t, i). \tag{13}$$

It is easy to see that $f$ given in eq. (13) is a solution in $C^{1,2}((0,\ T)\times\mathbb{R}\times S)$ to eqs (11) and (12) iff $g$ in $C^1((0, T)\times S)$ is a solution to

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t, i) + \inf_{u\in A}\Bigg[ -\theta(r + u_1(\hat{b} - r) + u_2(b(i) - r))g(t, i) + \frac{\theta(\theta+1)}{2}(u_1^2\|\hat{\sigma}\|^2 + u_2^2\|\bar{\sigma}(i)\|^2 + 2u_1u_2\hat{\sigma}\cdot\bar{\sigma}(i))g(t, i) + \lambda_i((1-u_2)^{-\theta} - 1)g(t, i) + \sum_{j\in S}\pi_{ij}(1-u_2)^{-\theta} g(t, j)\Bigg] = 0\,. \tag{14}$$

$i = 1, 2, 3, ...., M$ with terminal condition.

$$g(T, i) = 1. \tag{15}$$

Now we rewrite eq. (14) in the vector form.

Let $\bar{g}(t) = (g(t,\ 1),\ g(t,\ 2), \ldots, g(t,\ M))$, $I = I_{M\times M}$, $\bar{b} = \mathrm{diag}(b(1) - r,\ b(2) - r, \ldots, b(M) - r)$, $\bar{\lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M)$ and $\bar{\sigma}_u = \mathrm{diag}(\sigma_1^u, \sigma_2^u, ..., \sigma_M^u)$, where $\sigma_i^u = u_1^2\|\hat{\sigma}\|^2 + u_2^2\|\bar{\sigma}(i)\|^2 + 2u_1u_2\hat{\sigma}\cdot\bar{\sigma}(i)$. Then eq. (14) reduces to the form

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}t}(t) + \inf_{u\in A}\Bigg[ -\theta\{u_2\bar{b} + [r + u_1(\hat{b} - r)]I\}\bar{g}(t) + \frac{\theta(\theta+1)}{2}\bar{\sigma}_u\bar{g}(t) + ((1-u_2)^{-\theta} - 1)\bar{\lambda}\bar{g}(t) + (1-u_2)^{-\theta}\Pi\bar{g}(t)\Bigg] = 0. \tag{16}$$

This implies

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}t}(t) + \inf_{u\in A} B(u)\bar{g}(t) = 0, \tag{17}$$

with terminal condition

$$\bar{g}(T) = \overbrace{(1, 1, ..., 1)}^{M \text{ times}}, \tag{18}$$

where $B(u) := (b_{ij}(u))_{M\times M}$,

$$b_{ij}(u) = \begin{cases} -\theta(r + u_1(\hat{b} - r) + u_2(b(i) - r)) + \dfrac{\theta(\theta+1)}{2}\sigma_i^u - \lambda_i, & \text{if } i = j, \\ \lambda_i(1-u_2)^{-\theta} p_{ij}, & \text{if } i \neq j. \end{cases}$$

**Theorem 2.** *The value function defined by eq. (10) is the unique positive $C^{1,2}((0, T)\times\mathbb{R}\times S)$ solution of the eqs (11) and (12). Moreover, if $u^*(t, i)$ minimizing selector in eq. (14), then $u^*(t, Y(t-))$ is an optimal control.*

**Proof.** Consider the function

$$F(\bar{y}) = \inf_{u\in A} B(u)\bar{y},\ \ \bar{y}\in\mathbb{R}_+^M.$$

Then $F(\cdot)$ is Lipschitz continuous because $A$ is compact. Hence eqs (17) and (18) have a unique solution in $C^1((0, T)\times S)$. Therefore, it follows that the eqs (11) and (12) have a unique solution in $C^{1,2}((0,\ T)\times\mathbb{R}\times S)$. Let $u(\cdot)$ be an admissible control and $X(\cdot)$ be the corresponding wealth process eq. (3). Applying Ito's formula (see ref. 9, theorem 33, pp. 81–82) to $f(t, X(t), Y(t))$, we get

$$f(T, X(T), Y(T)) - f(0, X(0), Y(0)) = \int_0^T f_t(t, X(t-), Y(t-))\mathrm{d}t + \int_0^T f_x(t, X(t-), Y(t-))\mathrm{d}X^c(t)$$

$$+\frac{1}{2}\int_0^T f_{xx}(t, X(t-), Y(t-))\mathrm{d}[X, X]^c(t)$$

$$+\sum_{0<t\le T}\{f(t, X(t), Y(t)) - f(t, X(t-), Y(t-))\}, \quad (19)$$

where $[X, X]^c(t)$ denotes the continuous part of $[X, X](t)$ and $X^c(t)$ denotes the continuous part of $X(t)$. Taking expectation we get

$$E\{f(T, X(T), Y(T)) - f(0, X(0), Y(0))\}$$

$$= E\int_0^T [f_t(t, X(t-), Y(t-)) + X(t-) f_x(t, X(t-), Y(t-))$$

$$\times \{r + u_1(t)(\hat{b} - r) + u_2(t)(b(Y(t)) - r)\}$$

$$+\frac{1}{2}X(t-)^2 f_{xx}(t, X(t-), Y(t-))$$

$$\times (u_1^2(t)\|\hat{\sigma}\|^2 + u_2^2(t)\|\bar{\sigma}(Y(t-))\|^2$$

$$+ 2u_1(t)u_2(t)\hat{\sigma}\cdot\bar{\sigma}(Y(t-)))]\mathrm{d}t$$

$$+E\int_0^T\int_{\mathbb{R}}\{f(t, (1-u_2(t-))X(t-), Y(t-) + h(Y(t-), z))$$

$$- f(t, (1-u_2(t-))X(t-), Y(t-))\}m(\mathrm{d}z)\mathrm{d}t$$

$$+E\int_0^T\{f(t, (1-u_2(t-))X(t-), Y(t-))$$

$$- f(t, X(t-), Y(t-))\}\lambda(Y(t-))\mathrm{d}t. \quad (20)$$

Taking conditional expectation condition on $X(0) = x$, $Y(0) = i$ we get

$$f(0, x, i) \le E^{u(\cdot)}[(X(T))^{-\theta} | X(0) = x, Y(0) = i],$$

which is true for all $u(\cdot)$. This implies that

$$f(0, x, i) \le \inf_{u(\cdot)\in\mathcal{A}} E^{u(\cdot)}[(X(T))^{-\theta} | X(0) = x, Y(0) = i], \quad (21)$$

Now, we are going to prove that if $u^*(t, i)$ is the minimizing selector in eq. (14), then $u^*(t, Y(t-))$ is an optimal control. Observe that $\hat{u}(t) = u^*(t, Y(t-))$ is a prescribed control.

Let $g^*(\cdot,\cdot)$ be the unique solution of eqs (14) and (15) and $u^*(\cdot,\cdot) = (u_1^*(\cdot,\cdot), u_2^*(\cdot,\cdot))$ is a minimizing selector in eq. (14), then $g^*(\cdot,\cdot)$ is the unique solution to

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t, i) - \theta(r + u_1^*(t, i)(\hat{b} - r) + u_2^*(t, i)(b(i) - r))g(t, i)$$

$$+\frac{\theta(\theta+1)}{2}((u_1^*(t, i))^2\|\hat{\sigma}\|^2 + (u_2^*(t, i))^2\|\bar{\sigma}(i)\|^2$$

$$+ 2u_1^*(t, i)u_2^*(t, i)\hat{\sigma}\cdot\bar{\sigma}(i))g(t, i)$$

$$+ \lambda_i((1 - u_2^*(t, i))^{-\theta} - 1)g(t, i)$$

$$+\sum_{j\in S}\pi_{ij}(1 - u_2^*(t, i))^{-\theta} g(t, j) = 0, \quad (22)$$

with terminal condition

$$g(T, i) = 1. \quad (23)$$

Now we consider the equation

$$\frac{\partial f}{\partial t}(t, x, i) + x(r + u_1^*(t, i)(\hat{b} - r)$$

$$+ u_2^*(t, i)(b(i) - r))\frac{\partial f}{\partial x}(t, x, i)$$

$$+\frac{1}{2}x^2((u_1^*(t, i))^2\|\hat{\sigma}\|^2 + (u_2^*(t, i))^2\|\bar{\sigma}(i)\|^2$$

$$+ 2u_1^*(t, i)u_2^*(t, i)\hat{\sigma}\cdot\bar{\sigma}(i))\frac{\partial^2 f}{\partial x^2}(t, x, i)$$

$$+\sum_{j\in S}\pi_{i,j} f(t, (1 - u_2^*(t, i))x, j)$$

$$+ \lambda_i\{f(t, (1 - u_2^*(t, i))x, i) - f(t, x, i)\} = 0, \quad (24)$$

with terminal condition

$$f(T, x, i) = x^{-\theta}. \quad (25)$$

Let $\hat{f}(t, x, i)$ be the unique solution of eqs (24) and (25). Then, it is easy to check that $\hat{f}(t, x, i) = x^{-\theta} g^*(t, i)$, where $g^*(t, i)$ is the unique solution of eqs (22) and (23). We already know that $f(t, x, i) = x^{-\theta}g^*(t, i)$, where $f(t, x, i)$ is the unique solution of eqs (11) and (12). Hence

$$\hat{f} = f. \quad (26)$$

Using Ito's formula and eq. (24), we have

$$\hat{f}(t, x, i) = I_{\theta,T}(t, x, i, \hat{u}(\cdot)), \quad (27)$$

where $\hat{u}(t) = u^*(t, Y(t-))$, $t \ge 0$. From eqs (21), (26) and (27), we get

$$\inf_{u(\cdot)\in\mathcal{A}} I_{\theta,T}(t, x, i, u(\cdot)) \ge I_{\theta,T}(t, x, i, \hat{u}(\cdot)),$$

i.e.

$$\inf_{u(\cdot)\in\mathcal{A}} I_{\theta,T}(t, x, i, u(\cdot)) = I_{\theta,T}(t, x, i, \hat{u}(\cdot)).$$

Hence, $\hat{u}(\cdot)$ given by $\hat{u}(t) = u^*(t, Y(t-))$, $t \ge 0$ is an optimal control.

## Benchmarked asset management

In this section, we prove the existence of optimal strategy for the benchmarked problem described earlier. It suffices to consider the following payoff criterion given by

$$E[\exp(-\theta F(T))\,|\,F(0)=z,\,Y(0)=i]. \tag{28}$$

Let

$$\mathcal{I}_{\theta,T}(t,z,i,u(\cdot)) := E[\exp(-\theta F(T))\,|\,F(t)=z,\,Y(t)=i],$$

and

$$\psi(t,z,i) := \inf_{u(\cdot)\in\mathcal{A}} \mathcal{I}_{\theta,T}(t,z,i,u(\cdot)). \tag{29}$$

The HJB equation corresponding to the value function $\psi$ is given by

$$\frac{\partial\psi}{\partial t}(t,z,i)+\inf_{u\in A}\Bigg[\Bigg\{(r+u_1(\hat{b}-r)\mathrm{d}t+u_2(b(i)-r)) -\frac{1}{2}\|u_1\hat{\sigma}+u_2\bar{\sigma}(i)\|^2-\left(\alpha-\frac{1}{2}\|\bar{\gamma}\|^2\right)\Bigg\}\frac{\partial\psi}{\partial z}(t,z,i)$$

$$+\frac{1}{2}\|u_1\hat{\sigma}+u_2\bar{\sigma}(i)-\bar{\gamma}\|^2\frac{\partial^2\psi}{\partial z^2}(t,z,i)$$

$$+\sum_{j\in S}\pi_{ij}\psi(t,z+\ln(1-u_2),j)$$

$$+\lambda_i\{\psi(t,z+\ln(1-u_2),i)-\psi(t,z,i)\}\Bigg]=0, \tag{30}$$

with terminal condition

$$\psi(T,z,i)=\exp(-z\theta). \tag{31}$$

We look for a solution for eqs (30) and (31) of the form

$$\psi(t,z,i)=\exp(-z\theta)\varphi(t,i). \tag{32}$$

Then substituting eq. (32) in eq. (30), we obtain

$$\frac{\partial\varphi}{\partial t}(t,i)+\inf_{u\in A}\Bigg[-\theta\Bigg\{(r+u_1(\hat{b}-r)\mathrm{d}t+u_2(b(i)-r)) -\frac{1}{2}\|u_1\hat{\sigma}+u_2\bar{\sigma}(i)\|^2-\left(\alpha-\frac{1}{2}\|\bar{\gamma}\|^2\right)\Bigg\}\varphi(t,i)$$

$$+\frac{\theta^2}{2}\|u_1\hat{\sigma}+u_2\bar{\sigma}(i)-\bar{\gamma}\|^2\varphi(t,i)$$

$$+\lambda_i((1-u_2)^{-\theta}-1)\varphi(t,i)+\sum_{j\in S}\pi_{ij}(1-u_2)^{-\theta}\varphi(t,j)\Bigg]=0, \tag{33}$$

$i = 1, 2, 3, \ldots, M$ with terminal condition

$$\varphi(T,i)=1. \tag{34}$$

As in the proof of Theorem 2, we can show the following.

**Theorem 3.** *The value function defined by eq. (29) is the unique positive $C^{1,2}([0,T]\times\mathbb{R})$ solution of eqs (30) and (31). Moreover, if $u^*(t,i)$ is the minimizing selector in eq. (33), then $u^*(t,Y(t-))$ is an optimal control.*

## Numerical computation of optimal strategy

In this section we give a numerical method for computing optimal expected terminal utility function and corresponding optimal control for finite horizon problem described earlier. We have already seen that the system of ODEs (14) and (15) has a unique solution. From the eqs (17) and (18) it is clear that we can compute value function and optimal strategy by using numerical method for ODE. Here we use the Euler method.

Define

$$\hat{f}(t,\hat{g}(t))=\inf_{u\in A}B(u)\hat{g}(t). \tag{35}$$

Then we want to solve the system of ODEs given by

$$\frac{\mathrm{d}\hat{g}}{\mathrm{d}t}(t)=\hat{f}(t,\hat{g}(t)). \tag{36}$$

Let $\Delta t$ be the step size. At each step, with $\hat{g}(n\Delta t)$ available from the previous step, the explicit computational scheme is given by

$$\hat{g}((n+1)\Delta t)=\hat{g}(n\Delta t)+\Delta t\,\hat{f}(n\Delta t,\hat{g}(n\Delta t)),$$

with initial condition

$$\hat{g}(0)=(\overbrace{1,1,\ldots,1}^{M\text{ times}}).$$

At each step, when we compute $\hat{f}(n\Delta t,\hat{g}(n\Delta t))$, we have to first find the $u^*$ for which minimum will attain for eq. (35) and that is the optimal strategy.

To illustrate the results we next consider an example of Markov modulated market with three regimes for the time interval [0, 1]. We take maturity time $T = 1$ month. The time step length $\Delta t$ is taken as 0.001 month. The state space is $S = \{1, 2, 3\}$. The drift coefficient and volatility rate at each regime are chosen as follows:

$$(b(i), \sigma(i)) = \begin{cases} (0.65, 0.4), & \text{if } i = 1, \\ (0.5, 0.35), & \text{if } i = 2, \\ (0.4, 0.3), & \text{if } i = 3. \end{cases}$$

The transition probability matrix is assumed to be given by

$$(p_{ij}) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

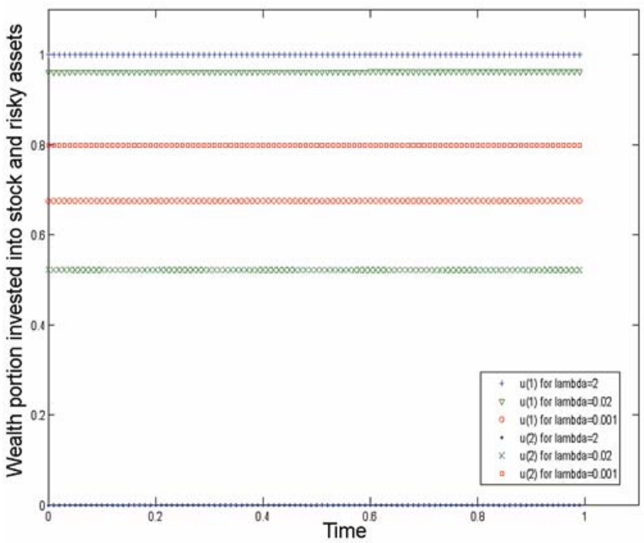


**Figure 1.** Optimal portfolio strategy functions: $u(1)$, Wealth portion invested into stock. $u(2)$, Wealth portion invested into defaultable asset.

**Table 1.** Optimal portfolio strategy for $\lambda = 0.02$

| Time | 0.0990 | 0.2990 | 0.4990 | 0.6990 | 0.8990 | 0.9990 |
|---|---|---|---|---|---|---|
| $u(1)$ | 0.9624 | 0.9644 | 0.9659 | 0.9686 | 0.9692 | 0.9694 |
| $u(2)$ | 0.5216 | 0.5204 | 0.5195 | 0.5195 | 0.5189 | 0.5186 |

**Table 2.** Optimal portfolio strategy for $\lambda = 0.001$

| Time | 0.0990 | 0.2990 | 0.4990 | 0.6990 | 0.8990 | 0.9990 |
|---|---|---|---|---|---|---|
| $u(1)$ | 0.6755 | 0.6760 | 0.6763 | 0.6762 | 0.6739 | 0.6750 |
| $u(2)$ | 0.7987 | 0.7980 | 0.7980 | 0.7997 | 0.8007 | 0.8002 |

The drift coefficient and volatility rate of the stock are $\hat{b} = 0.3$ and $\hat{\sigma} = 0.2$ respectively, bank interest rate $r = 0.15$, risk aversion parameter $\theta = 1$ and default intensity $(\lambda_1, \lambda_2, \lambda_3) = (2, 0.02, 0.001)$.

For this Markov modulated market we compute the value function and optimal strategy using the numerical technique described above.

Figure 1 describes the effect of different values of default intensity in the optimal portfolio strategy. It is clear from the figure that an investor invests a small portion of his wealth into that defaultable asset which has large default intensity and invests a very large portion his wealth into stock. For very small default intensity of the defaultable asset, the investor invests a significant amount of his wealth into defaultable asset.

Tables 1 and 2 describe the value of optimal strategy against time with different default intensities 0.02 and 0.001 respectively. Here, $u(1)$ and $u(2)$ denote the fraction of wealth invested in the stock and the defaultable asset respectively. Since the step size $\Delta t = 0.001$ is very small, actual data give the values of $u(1)$ and $u(2)$ for 1,000 time-points. Due to lack of space, we only display $u(1)$ and $u(2)$ at some specific time-points.


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